COMPARING THEORETICALLY MODELED STIFFNESS WITH EXPERIMENTALLY MEASURED STIFFNESS USING GEOMETRIC INVARIANTS

By

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The study investigates primarily the validity of a proposed stiffness model of a 6-DOF in-parallel compliant mechanism (hexapod machine or Stewart Platform). The mechanism is a good example for demonstrating the theory of stiffness mapping to control the force and position for a robot manipulator. An analysis of stiffness mappings may be essential to assess the quality of force and position control for robot manipulators.

A 6-DOF in-parallel compliant mechanism is integrated with a GE-P60 robot to perform an experiment for the measurement of twist and wrench matrices. The experimental results are used to check the validity of the proposed theoretical stiffness model. A geometrically meaningful methodology is developed for this comparison. The method implements geometric invariants for comparing a pair of like geometric entities and is applied to active plane transformations and stiffness mappings. This study demonstrates that there are no more than twenty-four parameters available for comparing one stiffness mapping with another. When utilizing stiffness matrices using geometric invariants, the comparison involves four independent quantities: eigenvalues, screws' pitches, angles between screws, and perpendicular distances between screws. All of these quantities have distinct dimensions and are, therefore, incompatible. No factor can be assigned to scale all of them to yield a unique quantity with geometric meaning.

The equivalent stiffness matrices could be defined by means of equivalent wrench matrices according to the wrench effects when a twist matrix is applied to the compliant system. This definition characterizes the physical effect occurring in the given systems.

Geometric invariants, equivalent wrench effects, and the reciprocal product of a twist screw and a wrench screw are used to investigate the relationship between the theoretical and experimental stiffness mappings. The results show good correlation. A 6-DOC wrench control using the proposed theoretical stiffness model is performed to examine the capability of a 6-DOF in-parallel compliant mechanism in compliant motions. The proposed theoretical model was found to be reliable in predicting the performance characteristic (wrench and twist) of a 6-DOF in-parallel compliant mechanism.

CHAPTER 1
INTRODUCTION

The objective of this study is to experimentally investigate the validity of a proposed stiffness model of a 6-DOF in-parallel compliant mechanism. The specific application considered is Kinestatic Control [1] which is dependent upon the modeling stiffness mapping of a 6-DOF in-parallel compliant mechanism. As models are generated, means for quantifying (accessing) how well two given mappings agree needs to be established. To achieve this need, a geometrically meaningful method to quantify the agreement between a theoretical stiffness model and an experimental stiffness model is desired to be developed. A stiffness mapping is essentially a geometric mapping from a twist space to a wrench space.

A 6-DOF in-parallel compliant mechanism is a Stewart-type platform but with six independent compliant connectors. The Stewart platform was first introduced by Stewart [2] to be used as an aircraft simulator. This compliant mechanism was proposed by Griffis and Duffy [1,3] to be a compliant wrist attached between (1) the end-effector and (2) the end-link of a serial-chain robot manipulator for Kinestatic Control. In this study, the compliant mechanism is used to map a small infinitesimal twist between the two rigid bodies into the corresponding interactive wrench that acts between them. This concept is called "stiffness mapping," which relates twists with wrenches by a 6x6 stiffness matrix. The twist and wrench are associated with the screw algebra.

Ball [4] introduces Screw Theory and defines a screw as a line vector associated with a pitch. Basically, a twist in a twist space can be symbolized by screws, as can a wrench in a wrench space. These screws are represented by a sextuple of homogeneous coordinates. An introduction to Screw Theory and its application in the hybrid control of robotic manipulators can be found in References 3 and 5.

The development of successful strategies and implementations for controlling robot manipulators' force and motion during contact with the constrained surfaces requires maintaining the manipulator contact with the environment without damage to both the environment and the manipulator. It is also able to apply required forces to accomplish the task. The compliance of robot manipulators permits the application of an external constraint force to maintain the contact throughout a constrained motion task. This topic has attracted the intensive interest of a number of investigators [1,3,6-12].

The methods for the compliance motion control can be categorized into two basic types. The first, active compliance can be accomplished by using two approaches. One approach is to adjust joint-servos to obtain a linear relation between the force (or velocity) and displacement, such as damping control, stiffness control, and impedance control. The other approach is to control forces by controlling positions or controlling positions and forces together, such as compliant control, compliance and force control, and hybrid control. Secondly, passive compliance is made to incorporate a compliant device into the manipulator near its end-effector. One well-known example of passive compliance is the use of Remote-Center-Compliance (RCC) [10] to perform the peg-in-hole insertion. However, an RCC device was not appropriate for force control because of its high stiffness in axial displacement and cocking rotations. Active compliance offers the advantage of programmability, but an instability problem occurred on the highly rigid surface contact.

Passive compliance has the advantage of performance but degrades the position control ability. Nevertheless, it can react instantaneously to the impact loads usually caused by assembly or other constrained tasks and, therefore, to increase task execution speeds. The flexibility to accommodate geometric uncertainties and dimensional tolerances is also favored. The simple and real-time response avoids the complex controller and costly instrumentation system required in some industrial applications.

Griffis [1], and Griffis and Duffy [13] invented the Kinestatic Control for simultaneously regulating force and displacement. Griffis and Duffy [3] also introduced a class of simple compliant devices, which are classified as a 3-3 type of in-parallel mechanisms, for the control scheme. Kinestatic Control relies upon the modeling of stiffness mapping which is established by a 6-DOF in-parallel compliant mechanism. Griffis and Duffy [3] developed a scheme to model global stiffness mapping of simple compliant mechanism. The stiffness model is different from the one which was defined by Dimentberg [14] and used by the most previous works. For the 6-DOF in-parallel compliant mechanism, a theoretically modeled stiffness matrix is obtained by combining the one-dimensional $f_i=k_ix_i$ relations for each of the six compliant connectors with the geometry of the in-parallel mechanism. This combination results in a matrix that changes with the configuration (position and orientation) of the in-parallel mechanism.

The Kinestatic Control provides the methodology for the control of force and position, which accepts the stiffness of a robotic manipulator, but doesn't need to adjust a set of gains for each given task taken and chooses the location of a center-of-compliance as required by stiffness control [8,12]. This provides a great feature in the position and force control for a robot manipulator. However, the study of the stiffness characteristics of this class of compliant mechanisms is limited in the current literatures. Moreover, the experimental verification for this stiffness modeling is not available in References 1, 3, 13, and other literature.

This thesis adopts the stiffness model presented by Griffis and Duffy [1,3], and is an attempt to provide readers with a systematical scheme for the analysis of this class of in-parallel compliant mechanisms. For verifying the modeling, a prototype 6-DOF in-parallel mechanism is created for experimentation and the methodologies for comparing stiffness mappings are established in this study.

In this thesis, Chapter 2 provides readers with a systematic analysis of stiffness mapping characteristics for modeling this class of 6-DOF in-parallel compliant mechanism. This knowledge may also be beneficial in the development of a comparison of two stiffness mappings. It first gives an introduction to stiffness mappings and defines the meanings of correlation and collineation in projective geometry and in the stiffness mapping of a 6-DOF in-parallel compliant mechanism. The invariants of both mappings are discussed. The spatial mapping of screws is assumed to be a non-singular transformation throughout this study. Chapter 2 provides the derivation of a stiffness mapping of a special 6-6 in-parallel mechanism and gives a physical model. This model will be used to construct a prototype 6-DOF in-parallel compliant mechanism for the experiment in Chapter 3. The compliant characteristics of a 6-DOF in-parallel compliant mechanism can be understood by analyzing the geometric properties of its stiffness mapping through eigenvalue problems. Chapter 2 addresses a concise description of these characteristics, which include eigenscrews, eigenstiffnesses, twist- and wrench-compliant axes, rotation- and force-compliant axes, and principal screws.

For the same mechanism, an experimentally measured stiffness matrix can be obtained for some configuration by the direct external measurement of six independent twist and wrench pairs. However, this scheme results in a stiffness matrix that is valid only near the configuration (position and orientation of the mechanism) used for measurement. Chapter 3 introduces this experimental investigation of stiffness mappings. The experiment is implemented on the integration system of the 6-DOF in-parallel compliant mechanism and a GE-P60 robot manipulator. The mechanism is inserted as a compliant wrist between the last link and the gripper of the GE-P60 robot manipulator. The specific application of this installation is that the compliant mechanism can be used to model the stiffness of a serial-chain robot manipulator. The experiment is performed to measure twist matrices and wrench matrices for the 6-DOF in-parallel compliant mechanism. Thus, a twist matrix and a wrench matrix can be generated by moving the six individual actuators of the robot manipulator sequentially and recording the change in the force and torque. The stiffness matrix can be determined by the use of the measured twist matrix and wrench matrix. A six-degrees-of-constraint wrench control is performed to test the capability of a stiffness matrix in compliant motions in sec.3.5 and sec.3.6.

After the measurement, the verification of the experimental model and the theoretical model is desired. Chapters 4 and 5 are devoted to this issue.

Chapter 4 states the method of comparison of stiffness mappings using geometric invariants. The basic concepts comparing geometric entities in three-dimensional Euclidean space is addressed first. The introduction surveys the geometric entities, such as points, lines, planes and screws. A comparison of geometric mappings in Euclidean plane is given as the preliminary study of comparison using geometric invariants. For the case to compare two stiffness matrices, the comparison focuses on the invariant properties of eigenscrews and eigenstiffnesses of stiffness matrices. This concept provides a geometric basis for verifying two stiffness models.

In Chapter 5, an analysis of the experimental data is presented. The experimental data are obtained form the measurement of twist and wrench matrices as shown in Chapter 3. This investigates the effect of error propagation on the determination of a stiffness matrix due to the subjected twist error and wrench error. A new concept of equivalent stiffness matrices is developed by the means of equivalent wrench matrices. This concept is derived from the observation of the wrench response in the compliant system when a twist is applied. The verification of two stiffness models is demonstrated by applying the concept of equivalent wrench matrices. The comparison of wrench matrices is conducted by using the equivalent wrench effect, geometric invariants, and the reciprocal product of a twist screw and a wrench screw. The results are discussed.

Summary and conclusions, remarks about the study and future work, comprise Chapter 6.

Appendix A is a list of the experimental data collected from the measurement of twists and wrenches. The data are provided for readers reference.

Another possible method to compare the given stiffness mappings is introduced in Appendix B. The idea is to change the stiffness mappings' basis to result in stiffness mappings having uniform dimension. This transition is needed because a matrix norm cannot be defined for a matrix whose elements do not have the same dimension (for example, force and force/length, etc.). The new basis can be chosen from the six line-vectors along the connectors of a 6-DOF in-parallel compliant mechanism. This requires the mappings to be represented in the same base and to have the same referring point. A quantity considered as the potential energy of the system might be yielded from the expression. In this investigation, it is found that the potential energy of a system cannot be used as a metric. If the two stiffness matrices are equivalent, the potential energy computed from the two matrices would be equal. However, the two systems having the same potential energy do not represent that the stiffness matrices of the two systems are equivalent. The potential energy can only be used as an auxiliary condition for the comparison of two stiffness mappings.

CHAPTER 2
THEORETICALLY MODELING STIFFNESS MAPPING OF A 6-DOF IN-PARALLEL COMPLIANT MECHANISM

2.1 Introduction to Stiffness Mapping

A one-dimensional linear elastic deformation, for which stress and strain are related by Hook's law, i.e. stress = constant × strain, is the oldest and simplest case of a stiffness mapping. Consider now a six-dimensional case, a 6-DOF in-parallel compliant mechanism (6-DOF-IPCM)[1] which is a device that combines the one-dimensional linear deformation relations for each of the six compliant connectors with the geometry of a Stewart-type platform. This arrangement has different interpretations in Physics, Linear Algebra, and Projective Geometry.

In Physics, a relation between an infinitesimal twist and a finite wrench can be established by a 6x6 matrix, called a stiffness matrix. In Linear Algebra, a special type of function $\boldsymbol{S}\colon \boldsymbol{D} \rightarrow \boldsymbol{W}$, between a twist vector space $\boldsymbol{D}$ and a wrench vector space $\boldsymbol{W}$, is called stiffness mapping. The mapping is represented by a 6x6 matrix. A twist or a wrench can be represented as a screw. Ball [4] defined a screw as a line with a pitch in three-dimensional space. He also considered a screw to be a point in a projective five-dimensional space.

Thus, in terms of Projective Geometry, the stiffness mapping may be considered as a projective transformation of a five-dimensional space, $\mathcal{P}_5$. The mapping is a transformation of screws that transforms a twist into a wrench from axis to ray coordinates, or from ray to ray coordinates. The former is a mapping that transforms a point onto a four-dimensional subspace, which is a hyperplane of four dimensions. Such a mapping that transforms a point in $\mathcal{P}_5$ onto a hyperplane in $\mathcal{P}_5$ is a projective correlation. The latter is a mapping that transforms an element in $\mathcal{P}_5$ into an element of the same type. Such a mapping is a projective collineation (for example, point to point, hyperplane to hyperplane).

---

[1] Hereafter 6-DOF in-parallel compliant mechanism is noted as 6-DOF-IPCM for short.

The two mapping representations each have their own geometric foundations in the theory of kinematics. The correlation establishes a potential energy-based inner product for twist, while collineation defines the invariant of the "eigenstiffnesses" and "eigenscrews" for the mapping. Collineations form a group of transformations under the operation of composition or matrix multiplication. The product of two correlations is a collineation, and hence correlations do not possess the property of closure and thus do not form a group. The recognition of the two representations is important for us to apply the theory of stiffness mapping correctly to the motion and position control for robot manipulators.

The stiffness of a robot manipulator can be modeled by introducing a 6-DOF-IPCM, which establishes a stiffness mapping for the manipulator, as a compliant wrist. By using stiffness mapping, Griffis and Duffy [13] proposed the theory of Kinestatic Control for the simultaneous control of force and position for a robot manipulator's gripper while solely controlling the position and orientation of the end link. This inspiring work is notable for its geometric insight and the great contributions in the force and position control. Basically, a 6-DOF-IPCM was initiated from an elastically supported rigid body, which is a rigid body suspended in space by six elastic links or springs. A typical study about this system was originated from Ball [4].

Ball first implemented screw theory to investigate the dynamics of this system in a potential field. He introduced the idea of the principal screw of inertia. If an instantaneous wrench change acted onto a rigid body induces a corresponding twist in the same axis, the screw is the "principal screw of inertia." A rigid body has the same number of principal screws of inertia as of degrees of freedom.

After Ball's studies, many researchers such as Dimentberg [14], Loncaric [15,16], Patterson and Lipkin [17-21], Griffis [1], Griffis and Duffy [3, 13] present several properties of stiffness mappings for similar compliant systems.

Dimentberg [14] adopts Ball's model in which the system remains near its unloaded configuration. The investigation results in a stiffness mapping that is expressed by a symmetric matrix. The system can be characterized by a set of six screws. The center-of-elasticity exists only when six screws form a pair of coincident perpendicular trihedrals.

Loncaric [15,16] presents definitions of "center-of-compliance" and "center-of-elasticity" with the use of a coordinate transformation to diagonalize off-diagonal blocks of the 6x6 compliance matrix (An inverse of stiffness is called compliance). It is found that center-of-compliance and center-of-elasticity are the same point if the off-diagonal block of a compliance matrix can be reduced to zero by a proper choice of coordinate frame.

Patterson and Lipkin [17-19] analyze the structure of compliant matrices following the Dimentberg's model by formulating the eigenvalue problem. They state that a compliance matrix has a full set of eigenvalues, three positive and three negative, and the corresponding eigenscrews have three positive and three negative eigenpitches. The eigenvalues and the corresponding eigenpitches have the same sign. The same results are also claimed by Griffis and Duffy [13] in their study.

Griffis and Duffy [13] recognizes the distinction between the correlation and collineation form of stiffness mappings to investigate their invariant properties. Their investigations conclude that the eigenstiffnesses and eigenscrews of the collineation form of a stiffness mapping are invariant with a change of coordinate system. Nevertheless, it is not true for the correlation form of a stiffness mapping because there is no meaning for its eigenscrews and eigenstiffnesses.

It should be noted that Schwartz and Doty [22] observe invariant properties of the eigenvalue problem differently and find that the eigenvalue problem of a matrix is physically consistent when its diagonal elements have the same physical units and all its eigenvalues have those same units. The results are consistent with the analysis of a collineation form of stiffness mapping.

Beside these properties, Dimentberg [14] introduces a type of eigenvalue problem to generalize the "wrench-compliant axes." Patterson and Lipkin [17,19] add the properties of twist-compliant axes by formulating a different type of eigenvalue problem and conclude that a unique point, center-of-elasticity, exists. They point out that a center-of-compliance does not always exist, but the center-of-elasticity does. A compliant axis exists if a compliant system has a force-compliant axis and a rotation-compliant axis that are collinear. Center of elasticity is the unique point where the principal screws of twist- and wrench-compliant axes intersect. Those studies have great significance to the force and position control in robotics since one can model and match these characteristics to the task being performed.

In these investigations done by Dimentberg [14], Loncaric [15,16] , Patterson and Lipkin [17-19] the stiffness or compliant matrix is recognized as a 6x6 symmetric matrix. Griffis and Duffy [1] has extended the foundation of the proposed theory to identify that the stiffness matrix is symmetric only when the modeled compliant mechanism is in the unloaded position.

When subjected to external loads, the 6x6 stiffness matrix is not symmetric. Griffis and Duffy proposed a compliant mechanism, a class of 3-3 Stewart platform, for Kinestatic Control. They analyze the configuration of the mechanism and demonstrate that the stiffness matrix is a 6x6 asymmetric matrix with an upper-left 3x3 symmetric sub-matrix, when the mechanism is far away from its unloaded configuration. Ciblak and Lipkin [23] further show that the skew-symmetric part of a stiffness matrix is one-half the externally applied load expressed as a spatial cross product operator. These investigations appear to have profound consequences in stiffness mapping but seem to provide no have further analyses about its stiffness characteristics under this condition. However, these characteristics may be important issues to consider before applying such a mechanism to desired tasks.

In the remainder of this chapter, an attempt is made to provide a systematic analysis of the properties of the class of compliant mechanisms proposed by Griffis and Duffy. This work concerns the proposed model to attempt to analyze stiffness mappings from the symmetric case to asymmetric one. We first introduce the correlation and collineation forms of stiffness mapping for providing a fundamental knowledge about this theory. A conceptual development of a 6-DOF-IPCM is merged to provide basic modeling knowledge. The prototype compliant mechanism developed in Chapter 3 is thus based upon the model specified in this chapter. After these discussions, several important concepts about the geometric properties of stiffness mappings are investigated through the eigenvalue problems, which include eigenscrews, twist- and wrench-compliant axes, principal screws, and rotation- and force-compliant axes. A classification of stiffness matrices based upon the types of eigenscrew system is presented. A derivation of the existence of a compliant axis for a compliant mechanism is demonstrated. A compliant axis exists in the general compliant system at any position as long as its collineation form of stiffness mapping has two collinear eigensrcews with the eigenpitches having the same magnitude and opposite sign. This result provides the most general condition for the existence of a compliant axis for a 6-DOF-IPCM.

2.2 The Correlation and Collineation Forms of Stiffness Mappings

In the next two sections, the two representations of stiffness mapping, correlation and collineation forms, will be presented. The recognition to the difference between the two representations may be important on the application of stiffness mapping to the position and force control in robotics. The correlation form is illustrated first.

Correlation form

Suppose that ***D*** and ***W*** are two vector spaces over the real field ***R***. Let ***D*** be twist space, and ***W*** be a wrench space. A stiffness mapping from ***D*** to ***W*** is a function $\boldsymbol{S}: \boldsymbol{D} \to \boldsymbol{W}$. If $\delta\underline{D} \in \boldsymbol{D}$, $\delta\underline{w} \in \boldsymbol{W}$, [K] is the stiffness matrix for the mapping ***S*** (Hereafter $\boldsymbol{S}: \boldsymbol{D} \to \boldsymbol{W}$ is used to denote the stiffness mapping). Therefore, the stiffness mapping of a 6-DOF-IPCM is defined by a form,

$$\delta\underline{w} = [K]\delta\underline{D}. \tag{2.1}$$

$\delta\underline{D}=[\delta\underline{x}\ ;\ \delta\underline{\theta}]$ are the axial coordinates of an infinitesimal twist[2]. The two 3x1 vectors are the small translation $\delta\underline{x}$ and the small rotation angle $\delta\underline{\theta}$. $\delta\underline{w}=[\delta\underline{f}\ ;\ \delta\underline{m}_0]$ is a small wrench change that is represented by ray coordinates. The two 3x1 vectors are the small force change $\delta\underline{f}$ and the small torque change $\delta\underline{m}_0$. [K] is a 6x6 matrix which is specified by the configuration of a 6-DOF-IPCM. The transformation maps the axis change of a twist, $\delta\underline{D}$, into the corresponding line change of a wrench, $\delta\underline{w}$, i.e. from axis to ray. Therefore, the mapping is a correlation.

The invariant properties of this mapping are important to investigate in geometry. Consider a general Euclidean transformation. The ray coordinates $\delta\underline{w}=[\delta\underline{f};\delta\underline{m}_0]$ of a wrench are transformed by the relation,

---

[2] Here and throughout a screw in ray coordinates is expressed in lowercase characters and a screw in axis coordinates is expressed in uppercase characters. The notation, $[\delta\underline{x};\delta\underline{\theta}]$, is considered as a column vector and the transpose sign is overlooked.

$$\delta \underline{w} = [e]\delta \underline{w}', \tag{2.2}$$

where $\delta\underline{w}' = [\delta\underline{f}'; \delta\underline{m}_0']$ are ray coordinates of the wrench in the new coordinate system, and where the [e] is given by

$$[e] = \begin{pmatrix} R_3 & 0_3 \\ A_3 R_3 & R_3 \end{pmatrix}. \tag{2.3}$$

R_3 and A_3 represent the usual 3x3 rotation and 3x3 skew-symmetric translation matrices.

Under the same Euclidean coordinate transformation, the axial coordinates $\delta\underline{D} = [\delta\underline{x}; \delta\underline{\theta}]$ of a twist are transformed by a similar relation,

$$\delta \underline{D} = [E]\delta \underline{D}' \tag{2.4}$$

where $\delta\underline{D}' = [\delta\underline{x}'; \delta\underline{\theta}']$ are the axial coordinates of the twist expressed in the new coordinate system, and where the [E] is given by

$$[E] = \begin{pmatrix} R_3 & A_3 R_3 \\ 0_3 & R_3 \end{pmatrix}. \tag{2.5}$$

The transformation matrices [E] and [e] are related by

$$[E]^T = [e]^{-1}. \tag{2.6}$$

Substituting (2.2), (2.4) into (2.1) and inverting [e] yields $\delta\underline{w} = [e]^{-1}[K][E]\delta\underline{D}'$. Furthermore, substituting $[e]^{-1}$ for $[E]^T$ yields a new description of the same system,

$$\delta \underline{w}' = [E]^T [K][E]\delta \underline{D}'. \tag{2.7}$$

Hence, the representation of the stiffness mapping in the new coordinate system is

$$\delta \underline{w}' = [K']\delta \underline{D}'. \tag{2.8}$$

Comparing (2.8) with (2.7) obtains the form,

$$[K'] = [E]^T [K][E]. \tag{2.9}$$

From equation (2.9), it is known that the matrices [K] and [K'] are congruent and thus have the same invariant signature (π, υ, ζ), which is defined as the number of positive eigenvalues, π, the number of negative eigenvalues, υ, and the number of zero eigenvalues, ζ. The matrices [K] and [K'], however, do not yield the same eigenvalues[3] and eigenvectors under a Euclidean coordinate transformation.

In this demonstration, it is important to realize that the correlation form of a stiffness mapping under a Euclidean coordinate transformation results the relation of congruence that preserves its signature only (see also Ref.1).

Several geometric properties are established for the correlation form. Let $\underline{D}_1$, $\underline{D}_2$, and $\underline{D}_3$, be the axial homogeneous coordinates of twists, and for all $\underline{D}_1$, $\underline{D}_2$, and $\underline{D}_3 \in \boldsymbol{R}^6$, where $\boldsymbol{R}^6$ denotes a six-dimensional real vector space. A real inner product space on $\boldsymbol{R}^6$ for the axial coordinates of twists is defined as

$$<\underline{D}_1, \underline{D}_2> = \underline{D}_1^{\,T}[K]\underline{D}_2 \ , \tag{2.10}$$

which may satisfy the following:

(i) $<\underline{D}_1,\underline{D}_2> = <\underline{D}_2,\underline{D}_1>$,

(ii) $<\rho\underline{D}_1,\underline{D}_2> = \rho<\underline{D}_1,\underline{D}_2>$, and $<\underline{D}_1,\rho\underline{D}_2> = \rho<\underline{D}_1,\underline{D}_2>$ for all ρ,

(iii) $<\underline{D}_1+\underline{D}_2,\underline{D}_3> = <\underline{D}_1,\underline{D}_3> + <\underline{D}_2,\underline{D}_3>$,

---

[3] Strictly speaking, there is no meaning for the eigenvalues and eigenvectors for the stiffness mapping in its correlation form simply because it is not a mapping of a space onto itself. Such values and vectors are however commonly referred to as eigenvalues and eigenvectors in the literature.

(iv) $<\underline{D}_1,\underline{D}_1>\geq 0$, and $<\underline{D}_1,\underline{D}_1>=0$ only if $\underline{D}_1=\underline{0}$.

Property (i) is satisfied if and only if the matrix [K] is symmetric, and it is known as symmetry. Property (iv) is referred to as the positive-definite property of [K]. Properties (ii) and (iii) hold as bi-linearity of [K]. This finite-dimensional real inner product space is called a Euclidean space. These properties are conventionally found in analyzing the characteristics of the stiffness mapping when the system is unloaded, i.e., the stiffness matrix is symmetric.

There is a result established by the inner product of a differential twist. Consider any differential twist of a 6-DOF-IPCM. The potential energy of the spring system is a function of linear and angular displacements of the spring system. The inner product of the differential twist defines the second derivative of potential energy of the system at the instance (see Refs.16 & 24). For a stiffness mapping, the potential energy in the system is a local minimum for all incremental displacements when [K] has six positive eigenvalues [24]. The system in this configuration is stable.

The correlation form of stiffness mappings naturally defines a potential energy-based inner product for twist as defined in equation (2.10). The form is equivalent to the virtual work done by a twist and a wrench. Assume that a system has an instantaneous twist $\underline{D}$ and is subjected to a wrench $\underline{w}$. The virtual work (denoted as v.w) is given by

$$v.w=\underline{w}^T\underline{D}. \tag{2.11}$$

We now change the representation of a twist, such that $\underline{D}=[\Delta]\underline{d}$, where

$$where \quad [\Delta]=\begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}. \tag{2.12}$$

$[\Delta]$ is a 6x6 matrix, I_3 is a 3x3 identity matrix, and 0_3 is a 3x3 zero matrix. $[\Delta]$ is a operator which transforms a screw in axis coordinates to one in ray coordinates.

Applying D= [Δ]d, to equation (2.11), the yielded result is denoted by "r.p" such that,

$$r.p = \underline{w}^T[\Delta]\underline{d}, \tag{2.13}$$

Equation (2.13) is a form of the reciprocal product of two screws in ray coordinates. Two screws are reciprocal if their reciprocal product is zero. Using equation (2.11) and (2.13), we obtain the properties of a wrench and a twist: if the reciprocal product of a wrench and a twist is zero, they are reciprocal and the virtual work is zero. The reciprocal product of two screws is an important concept to establish the decomposition of the twists of freedom and the wrenches of constraint for the system and its contact with a working environment in force and position control. Lipkin and Duffy identified this importance in References 6 and 25.

Lipkin and Duffy introduce the unique hybrid decomposition of the twist spaces and the constraint space that is of independent origin, basis and scale. A general wrench is a linear combination of impulsive wrench space and constraint wrench space. A general twist is a linear combination of freedom twist space and compliance twist space. If the freedom twist space is reciprocal to the constraint wrench space, it follows that the compliance twist space is reciprocal to impulsive wrench space. It is important to note that this result realizes the fallacious interpretation for the orthogonal complements in hybrid control (see Refs.25 and 26).

The sum of the number of freedoms and the number of constraints is always six, but the two bases may not always span a completed basis for six dimensional vector space. This phenomenon can be investigated by considering a block which is restricted to contact with an xy-plane (Figure 2.1). In this

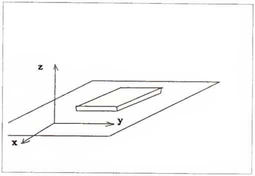

Figure 2.1 A block is only allowed to move along the xy-plane.

example, the twists of freedom $\underline{d}_f$ and wrenches of constraint w_c are

$$\underline{d}_f=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\\1&0&0\\0&1&0\\0&0&0\end{pmatrix},\ \underline{w}_c=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\\1&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

It can be shown that the two spaces spanned by $\underline{d}_f$ and $\underline{w}_c$ are reciprocal, but they do not span a completed six-dimensional vector space.

Collineation form

The stiffness mapping of a robot manipulator can be alternatively expressed as

$$\delta\underline{w} = [k]\delta\underline{d}, \tag{2.14}$$

where $\delta\underline{w}=[\delta\underline{f};\delta\underline{m}_0]$ and $\delta\underline{d}=[\delta\underline{\theta};\delta\underline{x}]$ are both in ray coordinates. The form (2.14) differs from the correlation form given by (2.1). Equation (2.14) is essentially a transformation of a line change of a small twist into the corresponding line change of a small wrench. The small change of the twist is an infinitesimal rotation about a line and has the homogeneous coordinates $\delta\underline{d}$. The small change of a wrench is an infinitesimal force change along a line and has the homogeneous coordinates $\delta\underline{w}$.

The relationship between [K] and [k] can be obtained by substituting $\delta\underline{D}=[\Delta]\delta\underline{d}$ into (2.1), which yields

$$\delta\underline{w}=[K][\Delta]\delta\underline{d}, \tag{2.15}$$

where $[\Delta]$ has been defined earlier in (2.12). The operator transforms a twist or a wrench in axis coordinates to one in ray coordinates. Therefore, $[\Delta]$ is a correlation. Comparing (2.15) with (2.14) obtains

$$[k]=[K][\Delta]. \tag{2.16}$$

This actually means that there are two successive correlation transformations of screws, where the first, [Δ] transforms ray to axis, and the second, [K] transforms axis to ray. Consequently, the overall transformation is the collineation form of stiffness mapping, i.e., ray to ray. Here, a screw is represented as a twist in twist space or as a wrench in wrench space. It is a one-to-one correspondence mapping that transforms an element onto an element of the same type. If we consider screws to be points in a projective 5-dimensional space as Ball [4] did, then a stiffness mapping transforms a point into a corresponding point in its dual space. This one-to-one correspondence between two points preserves all incidences.

There are two important statements that follow a projective collineation which is a projective transformation of points:

(i)If $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ are two collineations of S_5, then the composition or product of two collineations $\hat{\vartheta}_1\hat{\vartheta}_2$, a projective transformation with matrix AB, is also a collineation.

(ii)If $\hat{\vartheta}$ is a self-collineation of S_5, there are eigenpoints that are left invariant by $\hat{\vartheta}$. If $\hat{\vartheta}$ is the identical transformation I, every point of S_5 is an eigenpoint.

The invariant eigen-properties in (ii) can be examined by the following. Applying equation (2.3) for $\delta\underline{w}$ and $\delta\underline{d}$ yields $\delta\underline{w}=[e]\delta\underline{w}'$ and $\delta\underline{d}=[e]\delta\underline{d}'$. Substituting for (2.14) and inverting [e] yields

$$\underline{w}'=[e]^{-1}[k][e]\underline{d}' \tag{2.17}$$

which can be written as

$$\underline{w}'=[k']\underline{d}', \tag{2.18}$$

where

$$[k'] = [e]^{-1}[k][e]. \tag{2.19}$$

Equation (2.19) demonstrates that [k′] and [k] are similar matrices, which have the same eigenvalues. This shows that the eigenstiffnesses[4] of the collineation form of a stiffness mapping are invariant under a change of Euclidean coordinate system (see also Ref.1)

Since [k] and [k′] are related by a similarity transformation for a Euclidean coordinate transformation, the eigenscrews of the collineation form of stiffness mapping are also invariant. Although their representations change, the eigenvectors of [k] and [k′] locate the same screws in space under the change of Euclidean coordinate system. If an eigenscrew \$ is represented in two different coordinate systems by $\hat{s}$ and $\hat{s}'$ in ray coordinates, the two coordinate systems are related by,

$$\hat{s} = [e]\hat{s}'. \tag{2.20}$$

For a deeper understanding of the characteristics of eigenscrews , we consider the eigenscrews to be points in $\mathcal{S}_5$. The eigenpoints of a collineation ϑ of $\mathcal{S}_5$, with equation $\underline{z}' = A\underline{z}$, are obtained from the characteristic polynomial equation $|A - \lambda I| = 0$, where λ is characteristic value, and I is an identical matrix. To each of these λs there corresponds at least one vector $\underline{y}$, called an eigenvector of A, and every such vector gives the coordinates of an eigenpoint. The characteristics of the mapping can be investigated through those eigenpoints in the space. As in the case of a plane collineation, any given collineation transforms a range of points or pencil of planes into a homographically related range or pencil. A stiffness mapping is a non-singular 6x6 matrix which transforms any projective manifold, for example, a quadric or hyperplane-into another manifold of the same kind.

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[4] The eigenstiffness is the ratio of intensity of wrench to twist and is quantified by eigenvalue, whereas the eigenscrew is a screw that is mapped into itself by a stiffness mapping and whose homogenous coordinates are represented by an eigenvector.

2.3 Conceptual Design of A Compliant Mechanism

In this section, one shall introduce a conceptual development of a 6-DOF-IPCM first for readers to get an idea how to model this mechanism. This 6-DOF-IPCM is a class of Stewart-Platform and is derived from the type of 3-3 IPCM. Griffis and Duffy [3] introduced the stiffness mapping of a 3-3 IPCM for Kinestatic Control in their study. A 3-3 IPCM (an octahedron) is the simplest geometrical configuration of Stewart-type platform which has eight triangular faces. Its closed-form forward analysis involves the solution of the eighth degree polynomial [27]. This 3-3 IPCM is constructed by six connectors which meet in a pair-wise manner at three points in the top and base platforms. This structure, however, requires design of a pair of concentric ball and socket joints at each of the double connection points without mechanical interference in pair-wise connectors. This requirement make the design very complicated

A separate double connection point is preferred in the design, which leads to a special type of 6-6 in-parallel mechanism. The special 6-6 in-parallel mechanism was first developed by Griffis [28] for his work of Kinestatic Platform (an active type) for aerospace application. The mechanism possesses the advantage of reducing the complexity in the closed-form forward analysis [29,30] and the feature of six separated joint point locations in both the top of the platform and the base to avoid the mechanical interference problem. Hence, the special 6-6 in-parallel mechanism is implemented to construct the desired compliant device.

The geometry of the platform is defined by its twelve pivot points which are defined by points o, p, q, r, s, and t, with sub-labels 0 and 1 to distinguish bottom with top. Each connector is represented by connecting pivot points pair-wise from the base to the top. This arrangement is shown in Figure 2.2.

The geometric configurations of the base and the move platform are defined by two triangles. First, the points o, p, and q define an equi-angular, opq, and are called the

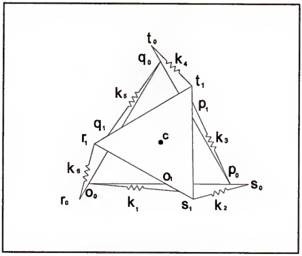
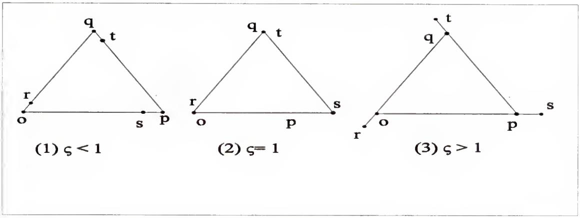


Figure 2.2 A plan view of the geometric configuration of a 6-DOF-IPCM. Each connector is represented by connecting pivot points pair-wise from the base to the top. The k_i is the stiffness constant for each connector.

vertexes of the equi-angular. Then points r, s, and t are determined by a scalar multiplier, ς, which is used to scale the length of line segments, os, pt, and qr on the sides of opq. A $\varsigma=1$ defines that the point s is coincident with the point p, t is coincident with q, and r is coincident with o. If $\varsigma<1$, the pivot point s is on the line op ad located between the pivot points o and p, point t between points p and q, and r between points q and o. Intuitively, when $\varsigma>1$, the length of line segment os is greater than op, and so forth. Figure 2.3 illustrates that the scalar multiplier, ς, is used to define the pivot points for the mechanism. Thus, the sizes of the base and the top platform can be scaled according the design specification. In Figure 2.2, the base has $\varsigma>1$, and the top platform has $\varsigma<1$.



(1) $\varsigma < 1$ (2) $\varsigma = 1$ (3) $\varsigma > 1$

Figure 2.3 A scalar multiplier ς is used to locate the position of the pivot points, r, s and t, on the sides, op, pq, and qo, of the equi-angular, opq. It is shown that (1) if $\varsigma < 1$, then os=pt=qr < op. (2) if $\varsigma=1$, then os=pt=qr=op. (3) if $\varsigma >1$, then os=pt=qr>op.

The size and the structural stability of the mechanism (for example, avoiding the singularity in the desired work space) are considered when one arranges the locations of the six pivot points. The structure stability can be investigated through Jacobian matrices. The forces F_p at the top platform relates the forces F_{cs} in the connectors of 6-DOF-IPCM with the form,

$$F_{cs}=j_c*F_p \tag{2.21}$$

where j_c is the 6x6 Jacobian matrix which is a linear transformation with mechanism's position and orientation. A singularity occurs when the determinant of j_c is zero. At this configuration, the ability to exert a load in certain direction is lost.

Examination of the desired mechanism is carried out by defining the geometric configuration of the mechanism and evaluating the determinant of the Jacobian matrix while varying position and orientation about the nominal position.

Each connector is modeled by two ball-and-socket joints and a translational spring. Therefore, it is considered as a spherical-prismatic-spherical (SPS) chain. Six connectors react to any external force applying to the platform and have linear displacements along the axial direction of each connector to maintain in static equilibrium. The translational spring needs to be carefully selected, because a well constructed 6-DOF-IPCM needs a certain rigid frame which does not deflect significantly under load, but still provides certain compliance.

2.4 Stiffness Matrix

Here, the stiffness matrix [K] of 6-DOF-IPCM (or a special 6-6 IPCM shown in Figure 2.4) is derived by using the scheme which was developed by Griffis and Duffy for analysis of a 3-3 IPCM [1,3]. Refer to Figure 2.5 which illustrates the plan view of the 6-

DOF-IPCM whose movable top platform is connected to the ground by six translation springs (considering as prismatic joints) acting in-parallel.

The derivation begins with the geometric definition of the mechanism. A coordinate system is selected to be fixed with the ground, with its origin at o_o. Points p_0, and s_0 are on the *x*-axis, while points q_0, r_0, and t_0 lie in the *xy* plane. Thus, the z-axis points right out of the paper as shown in Figure 2.5. All points are located relative to o_0 with position vectors represented by the sign '$\rightharpoonup$' at its top in the following equations.

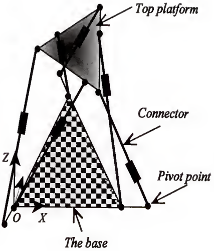


Figure 2.4 A 6-DOF-IPCM is a type of 6-6 Stewart-Platform.

Let the unloaded length of the six connectors be given and denoted by l_{0i}, where i=1, 2, 3,....,6. Furthermore, those distances between the six pivot points of the top, and the six pivot points of the bottom are all also known as constants.

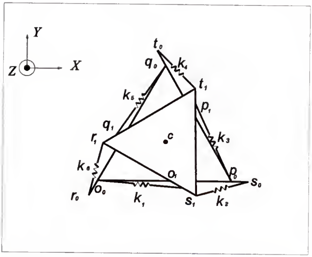


Figure 2.5 A plan view of a 6-6 in-parallel compliant mechanism. The coordinate system is shown in the illustration. The origin is located at the pivot point o_0.

Let the direction cosines of the line of the i-th connector be denoted by $\hat{s}_i$, and it can be obtained from position vectors aligned along a given connector. Thus, the orientations of its six connectors are defined by the following six position vectors whose magnitudes are the length of the six connectors:

$$\hat{s}_1=\frac{\overrightarrow{o_0o_1}}{l_1},\hat{s}_2=\frac{\overrightarrow{s_0s_1}}{l_2},\hat{s}_3=\frac{\overrightarrow{p_0p_1}}{l_3},\hat{s}_4=\frac{\overrightarrow{t_0t_1}}{l_4},\hat{s}_5=\frac{\overrightarrow{q_0q_1}}{l_5},\hat{s}_6=\frac{\overrightarrow{r_0r_1}}{l_6}. \tag{2.22}$$

where

$$l_1=\|\overrightarrow{o_0o_1}\|,\ l_2=\|\overrightarrow{s_0s_1}\|,\ l_3=\|\overrightarrow{p_0p_1}\|,\ l_4=\|\overrightarrow{t_0t_1}\|,\ l_5=\|\overrightarrow{q_0q_1}\|, l_6=\|\overrightarrow{r_0r_1}\|. \tag{2.23}$$

It is supposed that φ_i defines the elevation angles of the plane of the triangle, which are formed by the joint-pivots of the i-th connector with the next adjacent pivot point that shares the same side in the bottom plate of the mechanism, from the xy plane. The six triangles are defined as $o_0o_1p_0$, $p_0p_1q_0$, $q_0q_1o_0$, $s_0s_1o_0$, $t_0t_1p_0$, and $r_0r_1q_0$. An α_i angle defines the oriented angle of the i-th connector measured from the side, which lies on the xy plane, of the triangle. The six α_is are specified as follows:

$\alpha_1 = \angle\ o_1o_0s_0$, $\alpha_2 = \angle\ s_1s_0o_0$, $\alpha_3 = \angle\ p_1p_0t_0$, $\alpha_4 = \angle\ t_1t_0p_0$, $\alpha_5 = \angle\ q_1q_0r_0$, $\alpha_6 = \angle\ r_1r_0q_0$.

Assume that a resultant external wrench, having six homogenous coordinates $\hat{w}=[\underline{f};\underline{m}_0]$, is applied to the moveable platform. The resultant wrench is in static equilibrium with the six reacting spring forces, and the system remains in static equilibrium as the moveable platform twists relative to the ground. In order to accomplish this, the external wrench changes as the moveable platform is moving, and its change is defined as a wrench increment and has the six homogenous coordinates $\delta\hat{w}=[\delta f;\delta\underline{m}_0]$.

The reacting force of each spring system can be expressed by a line vector associating a force magnitude given by $\hat{w}_i = f_i[\hat{s}_i;\ \check{r}x\hat{s}_i]$, where f_i is force magnitude, $\hat{s}_i$ is the direction cosine of a line, and $\check{r}$ is the position vector from the origin to the line.

Thus, the total of the reaction forces of the six connectors caused by the resultant wrench is expressed as,

$$\begin{pmatrix} \underline{f} \\ \underline{m}_0 \end{pmatrix} = \begin{pmatrix} \hat{s}_1 & \hat{s}_2 & \hat{s}_3 & \hat{s}_4 & \hat{s}_5 & \hat{s}_6 \\ \hat{o} & \overrightarrow{o_0}s_o x\hat{s}_2 & \overrightarrow{o_0}p_0 x\hat{s}_3 & \overrightarrow{o_0}t_0 x\hat{s}_4 & \overrightarrow{o_0}q_0 x\hat{s}_5 & \overrightarrow{o_0}r_0 x\hat{s}_6 \end{pmatrix} \begin{pmatrix} k_1(l_1-l_{o1}) \\ k_2(l_2-l_{o2}) \\ k_3(l_3-l_{o3}) \\ k_4(l_4-l_{o4}) \\ k_5(l_5-l_{o5}) \\ k_6(l_6-l_{o6}) \end{pmatrix} \tag{2.24}$$

where $k_i(l_i\text{-}l_{oi})$ is the force in the i-th connector reacting to the extension of the i-th spring, where k_i is the spring stiffness of the i-th connector, and l_{oi} is the initial length of the i-th connector, and $\hat{o}$ is a zero vector. The upper three rows of the 6x6 matrix in (2.24) are the direction cosine of six lines. The lower three rows are the moments of six lines about the origin o_0. The 6x6 matrix is the Jacobian matrix which can be used to condition the status of the singular configuration. If the determinant is vanishing, it states the existence of a singularity in this configuration. Because the locations of all of the grounded pivots are known constants, the matrix is thus a function of the direction cosines of the six connectors.

The instantaneous change in the reaction force due to the applied wrench increment can be obtained by differentiating (2.24) with respect to l_i and $\hat{s}_i$; thus it yields,

$$\begin{pmatrix} \delta\underline{f} \\ \delta\underline{m}_0 \end{pmatrix} = \begin{pmatrix} \hat{s}_1 & \hat{s}_2 & \hat{s}_3 & \hat{s}_4 & \hat{s}_5 & \hat{s}_6 \\ \hat{o} & \overrightarrow{o_0}s_0 x\hat{s}_2 & \overrightarrow{o_0}p_0 x\hat{s}_3 & \overrightarrow{o_0}t_0 x\hat{s}_4 & \overrightarrow{o_0}q_0 x\hat{s}_5 & \overrightarrow{o_0}r_0 x\hat{s}_6 \end{pmatrix} \begin{pmatrix} k_1\delta l_1 \\ k_2\delta l_2 \\ k_3\delta l_3 \\ k_4\delta l_4 \\ k_5\delta l_5 \\ k_6\delta l_6 \end{pmatrix}$$

$$+\begin{pmatrix} \delta\hat{s}_1 & \delta\hat{s}_2 & \delta\hat{s}_3 & \delta\hat{s}_4 & \delta\hat{s}_5 & \delta\hat{s}_6 \\ \hat{o} & \overrightarrow{o_0s_0}x\delta\hat{s}_2 & \overrightarrow{o_0p_0}x\delta\hat{s}_3 & \overrightarrow{o_0t_0}x\delta\hat{s}_4 & \overrightarrow{o_0q_0}x\delta\hat{s}_5 & \overrightarrow{o_0r_0}x\delta\hat{s}_6 \end{pmatrix}\begin{pmatrix} k_1(1-\rho_1)l_1 \\ k_2(1-\rho_2)l_2 \\ k_3(1-\rho_3)l_3 \\ k_4(1-\rho_2)l_4 \\ k_5(1-\rho_5)l_5 \\ k_6(1-\rho_6)l_6 \end{pmatrix} \tag{2.25}$$

where $\rho_i = l_{0i}/l_i$, and $\delta\hat{s}_i$ is the derivative of the line coordinates of the i-th connector which remains to be determined.

The instantaneous change of the orientation of a representative i-th connector is the derivative with respect to φ_i and α_i. Figure 2.6 shows that the derivative of a representative of the i-th connector, $\delta\hat{s}_i$, which may be written as,

$$\delta\hat{s}_i = \delta\hat{s}_i^{\varphi} s_{\alpha i}\delta\varphi + \delta\hat{s}_i^{\alpha}\delta\alpha, \tag{2.26}$$

where $\delta\hat{s}^{\varphi}_i$ and $\delta\hat{s}^{\alpha}$ are unit vectors defined as

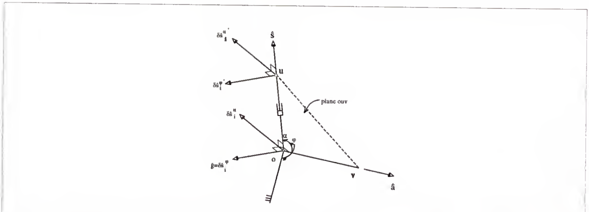

$\delta\hat{s}_i^{\alpha'}$
$\hat{s}$
u
$\delta\hat{s}_i^{\varphi'}$
plane ouv
$\delta\hat{s}_i^{\alpha}$
α
φ
o
$\hat{g}=\delta\hat{s}_i^{\varphi}$
v
$\hat{a}$

Figure 2.6 The line vectors $\hat{s}_i$, $\delta\hat{s}_i^{\varphi}$, $\delta\hat{s}_i^{\alpha}$ of the i-th connector of the 6-DOF-IPCM. The α defines the oriented angle of $\hat{s}_i$ measured from the side vector $\hat{a}$.

$$\delta\hat{s}_i^{\varphi} = \hat{g}_i, \quad and \;\; \delta\hat{s}_i^{\alpha} = \hat{g}_i x \hat{s}_i, \tag{2.27}$$

and where

$$\hat{g}_i = \frac{\hat{a}_i x \hat{s}_i}{\|\hat{a}_i x \hat{s}_i\|}. \tag{2.28}$$

The six $\hat{a}_i$s in (2.28) are constants given by,

$$\hat{a}_1 = \frac{\overrightarrow{o_0 s_o}}{\|\overrightarrow{o_0 s_o}\|},\ \hat{a}_2 = \frac{-\ \overrightarrow{o_0 s_o}}{\|\overrightarrow{o_0 s_o}\|},\ \hat{a}_3 = \frac{\overrightarrow{p_0 t_o}}{\|\overrightarrow{p_0 t_o}\|},\ \hat{a}_4 = \frac{-\ \overrightarrow{p_0 t_o}}{\|\overrightarrow{p_0 t_o}\|},\ \hat{a}_5 = \frac{-\ \overrightarrow{o_0 q_o}}{\|\overrightarrow{o_0 q_0}\|},\ \hat{a}_6 = \frac{\overrightarrow{o_0 q_o}}{\|\overrightarrow{o_0 q_o}\|}. \tag{2.29}$$

$s_{\alpha i}$ is the shorthand of $\sin(\alpha_i)$. Therefore, substituting (2.26) into (2.25) and distributing the terms yields the form,

$$\begin{pmatrix} \delta \underline{f} \\ \delta \underline{m}_0 \end{pmatrix} = \begin{pmatrix} \hat{s}_1 & \hat{s}_2 & \hat{s}_3 & \hat{s}_4 & \hat{s}_5 & \hat{s}_6 \\ \hat{o} & \overrightarrow{o_0 s_0} x \hat{s}_2 & \overrightarrow{o_0 p_0} x \hat{s}_3 & \overrightarrow{o_0 t_0} x \hat{s}_4 & \overrightarrow{o_0 q_0} x \hat{s}_5 & \overrightarrow{o_0 r_0} x \hat{s}_6 \end{pmatrix} [k_i] \begin{pmatrix} \delta l_1 \\ \delta l_2 \\ \delta l_3 \\ \delta l_4 \\ \delta l_5 \\ \delta l_6 \end{pmatrix}$$

$$+ \begin{pmatrix} \delta\hat{s}_1^{\varphi} & \delta\hat{s}_2^{\varphi} & \delta\hat{s}_3^{\varphi} & \delta\hat{s}_4^{\varphi} & \delta\hat{s}_5^{\varphi} & \delta\hat{s}_6^{\varphi} \\ \hat{o} & \overrightarrow{o_0 s_0} x \delta\hat{s}_2^{\varphi} & \overrightarrow{o_0 p_0} x \delta\hat{s}_3^{\varphi} & \overrightarrow{o_0 t_0} x \delta\hat{s}_4^{\varphi} & \overrightarrow{o_0 q_0} x \delta\hat{s}_5^{\varphi} & \overrightarrow{o_0 r_0} x \delta\hat{s}_6^{\varphi} \end{pmatrix} [k_i(1-\rho_i)] \begin{pmatrix} l_1 s_{\alpha 1} \delta\varphi_1 \\ l_2 s_{\alpha 2} \delta\varphi_2 \\ l_3 s_{\alpha 3} \delta\varphi_3 \\ l_4 s_{\alpha 4} \delta\varphi_4 \\ l_5 s_{\alpha 5} \delta\varphi_5 \\ l_6 s_{\alpha 6} \delta\varphi_6 \end{pmatrix}$$

$$+\begin{pmatrix} \delta\hat{s}_1^{\alpha} & \delta\hat{s}_2^{\alpha} & \delta\hat{s}_3^{\alpha} & \delta\hat{s}_4^{\alpha} & \delta\hat{s}_5^{\alpha} & \delta\hat{s}_6^{\alpha} \\ \hat{o} & \overrightarrow{o_0s_0}x\delta\hat{s}_2^{\alpha} & \overrightarrow{o_0p_0}x\delta\hat{s}_3^{\alpha} & \overrightarrow{o_0t_0}x\delta\hat{s}_4^{\alpha} & \overrightarrow{o_0q_0}x\delta\hat{s}_5^{\alpha} & \overrightarrow{o_0r_0}x\delta\hat{s}_6^{\alpha} \end{pmatrix}[k_i(1-\rho_i)]\begin{pmatrix} l_1\delta\alpha_1 \\ l_2\delta\alpha_2 \\ l_3\delta\alpha_3 \\ l_4\delta\alpha_4 \\ l_5\delta\alpha_5 \\ l_6\delta\alpha_6 \end{pmatrix} \tag{2.30}$$

or

$$\delta w=j[k_i]\delta\underline{l}+[\delta j_{\varphi}][k_i(1-\rho_i)]\delta\underline{\varphi}^*+[\delta j_{\alpha}][k_i(1-\rho_i)]\delta\underline{\alpha}^* \tag{2.31}$$

where the substitutions

$$j=\begin{pmatrix} \hat{s}_1 & \hat{s}_2 & \hat{s}_3 & \hat{s}_4 & \hat{s}_5 & \hat{s}_6 \\ \hat{o} & \overrightarrow{o_0s_0}x\hat{s}_2 & \overrightarrow{o_0p_0}x\hat{s}_3 & \overrightarrow{o_0t_0}x\hat{s}_4 & \overrightarrow{o_0q_0}x\hat{s}_5 & \overrightarrow{o_0r_0}x\hat{s}_6 \end{pmatrix},$$

$$[\delta j_{\alpha}]=\begin{pmatrix} \delta\hat{s}_1^{\alpha} & \delta\hat{s}_2^{\alpha} & \delta\hat{s}_3^{\alpha} & \delta\hat{s}_4^{\alpha} & \delta\hat{s}_5^{\alpha} & \delta\hat{s}_6^{\alpha} \\ \hat{o} & \overrightarrow{o_0s_0}x\delta\hat{s}_2^{\alpha} & \overrightarrow{o_0p_0}x\delta\hat{s}_3^{\alpha} & \overrightarrow{o_0t_0}x\delta\hat{s}_4^{\alpha} & \overrightarrow{o_0q_0}x\delta\hat{s}_5^{\alpha} & \overrightarrow{o_0r_0}x\delta\hat{s}_6^{\alpha} \end{pmatrix},$$

$$[\delta j_{\varphi}]=\begin{pmatrix} \delta\hat{s}_1^{\varphi} & \delta\hat{s}_2^{\varphi} & \delta\hat{s}_3^{\varphi} & \delta\hat{s}_4^{\varphi} & \delta\hat{s}_5^{\varphi} & \delta\hat{s}_6^{\varphi} \\ \hat{o} & \overrightarrow{o_0s_0}x\delta\hat{s}_2^{\varphi} & \overrightarrow{o_0p_0}x\delta\hat{s}_3^{\varphi} & \overrightarrow{o_0t_0}x\delta\hat{s}_4^{\varphi} & \overrightarrow{o_0q_0}x\delta\hat{s}_5^{\varphi} & \overrightarrow{o_0r_0}x\delta\hat{s}_6^{\varphi} \end{pmatrix},$$

$\delta\hat{w}=[\delta\underline{f},\delta\underline{m}_0]^T$, $\delta\underline{\varphi}^*=(l_1s_{\alpha 1}\delta\varphi_1, l_2s_{\alpha 2}\delta\varphi_2....,l_6s_{\alpha 6}\delta\varphi_6)^T$, $\delta\underline{l}=(\delta l_1 \;\; \delta l_2 \;\; \delta l_3 \;\; \delta l_4 \;\; \delta l_5 \;\; \delta l_6)^T$ and $\delta\underline{\alpha}^*=[l_1\delta\alpha_1,.....,l_6\delta\alpha_6]^T$ have been introduced. In (2.31), $[k_i]$ and $[k_i(1\text{-}\rho_i)]$ are 6x6 diagonal matrices, while j, $[\delta j_{\varphi}]$, and $[\delta j_{\alpha}]$ are 6x6 matrices. The i-th column of j has line coordinates of the i-th connector, $\hat{s}_i$; the i-th column of $[\delta j_{\varphi}]$ contains the derivative of the line coordinates of the i-th connector with respect to φ_i, let it be denoted by $\delta\hat{s}_i^{\varphi}$; the i-th column of $[\delta j_{\alpha}]$ represents the derivative of line coordinates with respect to α_i, and let it be denoted by $\delta\hat{s}_i^{\alpha}$.

$\delta \underline{l}$, $\delta \underline{\varphi}^*$ and $\delta \underline{\alpha}^*$ are 6x1 vectors which can be further defined. Let $\delta \underline{D}$ be denoted as the infinitesimal twist. The small length changes in connectors yields the expression,

$$\delta \underline{l} = j^T \ \delta \underline{D}, \tag{2.32}$$

which is the reverse kinematic solution (see Refs.1 and 5).

As shown in Figure 2.6, the line coordinates $\hat{s}_i$, $\delta \hat{s}_i^{\varphi}$, $\delta \hat{s}_i^{\alpha}$ for the i-th connector of a 6-DOF-IPCM. The line coordinates $\delta \hat{s}_i^{\varphi'}$ and $\delta \hat{s}_i^{\alpha'}$ are obtained by self-displacing $\delta \hat{s}_i^{\varphi}$ and $\delta \hat{s}_i^{\alpha}$ a distance l_i along $\hat{s}_i$. Then, for each element of the column matrix $\delta \underline{\varphi}^*$,

$$l_i s_{\alpha i} \delta \varphi_i = (\delta \hat{s}_i^{\varphi'})^T \ \delta \underline{D} = (\delta \hat{s}_i^{\varphi} \bullet \delta \underline{x}) + \ \vec{ou} x \delta \hat{s}_i^{\varphi} \bullet \delta \underline{\theta}, \tag{2.33}$$

which is the projection of displacement of an end point of the i-th connector onto $\delta \hat{s}_i^{\varphi'}$. The normalized line coordinates $\delta s^{\varphi'}$ are obtained by $\delta \hat{s}_i^{\varphi'} = [\delta \hat{s}_i^{\varphi} ; \vec{ou} x \delta \hat{s}_i^{\varphi}]$, where $\vec{ou}$ is a position vector from point o to point u (the end point of the i-th connector). Therefore,

$$\delta \underline{\varphi}^* = [\delta j_{\varphi}']^T \ \delta \underline{D}, \tag{2.34}$$

where

$$[\delta j_{\varphi}'] = \begin{pmatrix} \delta \hat{s}_1^{\varphi} & \delta \hat{s}_2^{\varphi} & \delta \hat{s}_3^{\varphi} & \delta \hat{s}_4^{\varphi} & \delta \hat{s}_5^{\varphi} & \delta \hat{s}_6^{\varphi} \\ \vec{o_0 o_1} x \delta \hat{s}_1^{\varphi} & \vec{o_0 s_1} x \delta \hat{s}_2^{\varphi} & \vec{o_0 p_1} x \delta \hat{s}_3^{\varphi} & \vec{o_0 t_1} x \delta \hat{s}_4^{\varphi} & \vec{o_0 q_1} x \delta \hat{s}_5^{\varphi} & \vec{o_0 r_1} x \delta \hat{s}_6^{\varphi} \end{pmatrix}, \tag{2.35}$$

where the i-th column of the 6x6 matrix $[\delta j_{\varphi}']$ are the normalized line coordinates of the line $\delta \hat{s}_i^{\varphi'}$. Since $\delta \hat{s}_i^{\varphi'}$ is a line obtained by self-parallel of the i-th line $\delta \hat{s}_i^{\varphi}$ a distance l_i along the i-th connector, $[\delta j_{\varphi}']$ may be further decomposed into

$$[\delta j_{\varphi}'] = [\delta j_{\varphi}] + [g_{\varphi}], \tag{2.36}$$

where

$$[g_{\varphi}] = \begin{pmatrix} \hat{o} & \hat{o} & \hat{o} & \hat{o} & \hat{o} & \hat{o} \\ l_1\hat{s}_1 x \hat{g}_1 & l_2\hat{s}_2 x \hat{g}_2 & l_3\hat{s}_3 x \hat{g}_3 & l_4\hat{s}_4 x \hat{g}_4 & l_5\hat{s}_5 x \hat{g}_5 & l_6\hat{s}_6 x \hat{g}_6 \end{pmatrix} \tag{2.37}$$

is the 6x6 matrix of moment vectors whose i-th column results from a self-parallel translation of the i-th line coordinates $\delta\hat{s}_i^{\varphi}$ a distance l_i along the i-th connector; $[\delta j_{\varphi}]$ is as defined before in (2.31).

Referring to Figure 2.6 again, for each element of the column matrix $\delta\underline{\alpha}^*$,

$$l_i\delta\alpha_i = (\delta\hat{s}_i^{\alpha'})^T\ \delta\underline{D} = (\delta\hat{s}_i^{\alpha}\bullet\delta x) + \ \vec{ou}\ x\ \delta\hat{s}_i^{\alpha}\bullet\delta\varphi, \tag{2.38}$$

which is the projection of displacement of an end point of the i-th connector onto $\delta\hat{s}_i^{\alpha'}$. The normalized line coordinates $\delta\hat{s}^{\alpha'}$ are obtained by $\delta\hat{s}_i^{\alpha'} = [\delta\hat{s}_i^{\alpha}; \vec{ou} x \delta\hat{s}_i^{\alpha}]$, where $\vec{ou}$ is a position vector from point o to point u (the end point of the i-th connector). Therefore,

$$\delta\underline{\alpha}^* = [\delta j_{\alpha}']\delta\underline{D}, \tag{2.39}$$

where

$$[\delta j_{\alpha}'] = \begin{pmatrix} \delta\hat{s}_1^{\alpha} & \delta\hat{s}_2^{\alpha} & \delta\hat{s}_3^{\alpha} & \delta\hat{s}_4^{\alpha} & \delta\hat{s}_5^{\alpha} & \delta\hat{s}_6^{\alpha} \\ \overrightarrow{o_0 o_1} x \delta\vec{s}_1^{\alpha} & \overrightarrow{o_0 s_1} x \delta\hat{s}_2^{\alpha} & \overrightarrow{o_0 p_1} x \delta\hat{s}_3^{\alpha} & \overrightarrow{o_0 t_1} x \delta\hat{s}_4^{\alpha} & \overrightarrow{o_0 q_1} x \delta\hat{s}_5^{\alpha} & \overrightarrow{o_0 r_1} x \delta\hat{s}_6^{\alpha} \end{pmatrix} \tag{2.40}$$

where the i-th column of the 6x6 matrix $[\delta j_{\alpha}']$ are the normalized line coordinates of the line $\delta\hat{s}_i^{\alpha'}$. Since $\delta\hat{s}_i^{\alpha'}$ is a line obtained by self-parallel of the i-th line $\delta\hat{s}_i^{\alpha}$, a distance l_i along the i-th connector, $[\delta j_{\alpha}']$ may be further decomposed into

$$[\delta j_\alpha{}'] = [\delta j_\alpha] + [g_\alpha], \tag{2.41}$$

where

$$[g_\alpha] = \begin{pmatrix} \hat{o} & \hat{o} & \hat{o} & \hat{o} & \hat{o} & \hat{o} \\ l_1\hat{g}_1 & l_2\hat{g}_2 & l_3\hat{g} & l_4\hat{g}_4 & l_5\hat{g} & l_6\hat{g}_6 \end{pmatrix} \tag{2.42}$$

is the 6x6 matrix of moment vectors whose i-th column results from a self-parallel translation of the i-th line coordinates $\delta\hat{s}_i^\alpha$ a distance l_i along the i-th connector.

Substituting (2.32), (2.34) and (2.39) into (2.31) and regrouping these terms yields a stiffness mapping which is similar to the one reported in References 1 and 2,

$$\delta\hat{w} = (j[k_i]j^T + [\delta j_\varphi][k_i(1-\rho_i)]([\delta j_\varphi]^T + [g_\varphi]^T) + [\delta j_\alpha][k_i(1-\rho_i)]([\delta j_\alpha]^T + [g_\alpha]^T))\delta\underline{D},$$

or

$$\delta\hat{w} = [K]\delta\underline{D}. \tag{2.43}$$

In Equation (2.43), [K] is defined as the “stiffness mapping” which transforms a infinitesimal twist, $\delta\underline{D}$, in axial coordinates into a wrench increment, $\delta\hat{w}$, in ray coordinates. Therefore, the mapping is a correlation.

The mapping [K] is asymmetric due to the two terms: $[\delta j_\varphi][k_i(1-\rho_i)][g_\varphi]^T$ and $[\delta j_\alpha][k_i(1-\rho_i)][g_\alpha]^T$. Since the $[g_\varphi]$ and $[g_\alpha]$ have their upper three rows zero, the summation of all matrix multiplications is a matrix which is asymmetric and has the upper-left 3x3 submatrix symmetric. This reduces the number of independent parameters of [K] to 33.

In general, [K] is not necessary to be singular when the Jacobian matrix j is singular. When the system is unloaded, $1-\rho_i=0$, therefore the last two terms in the equation (2.43) vanish; equation (2.43) reduces to

$$\delta\hat{w} = (j[k_i]j^T)\delta\underline{D}, \tag{2.44}$$

which is symmetric and determined by twenty-one independent parameters. Furthermore, this stiffness matrix is dependent upon the Jacobian matrix *j*, i.e., when *j* is singular, [K] is also singular.

2.5 A Model

The derivation of a 6-DOF-IPCM has been presented in a prior section. A geometric configuration of the 6-DOF-IPCM is also shown in Figure 2.5. Thus, the stiffness of the compliant mechanism is ready to calculate when its physical configuration is specified.

As shown in Figure 2.5, the 6-DOF-IPCM is defined by its spherical pivot points o, p, q, r, s, and t which are sub-labeled with 0 and 1 to distinguish the bottom and the top plates, respectively. A coordinate system Ω is fixed to the ground, with its origin at o_0. The orientation is the same as the global coordinate system. Points p_0, and s_0 are on the *x*-axis, while points q_0, r_0, and t_0 lie in the *xy* plane. All points are located relative to o_0.

$\|o_0p_0\|=\|p_0q_0\|=\|q_0o_0\|=4.500"$, and $\|o_0s_0\|=\|p_0t_0\|=\|q_0r_0\|=5.400"$ are lengths between pivot points on the bottom plate. $\|r_1s_1\|=\|s_1t_1\|=\|t_1r_1\|=4.500"$, and $\|r_1o_1\|=\|s_1p_1\|=\|t_1q_1\|=$ 3.600", are lengths between pivot points on the top plate. At its nominal position, the connector lengths for connectors $\|o_0o_1\|$, $\|p_0p_1\|$, $\|q_0q_1\|$ are 4.1232", and for connectors $\|s_0s_1\|$, $\|t_0t_1\|$, $\|r_0r_1\|$ are 3.5942", and the coordinates of the geometric center point c of the triangle $r_1s_1t_1$ are $(x_c, y_c, z_c)=(2.2500, 1.2990, 2.9745)$. The coordinates of the geometric center point c_0 of the triangle $o_0p_0q_0$ are $(x_{c0}, y_{cc0}, z_{c0})=(2.2500, 1.2990, 0.0)$.

In this investigation, all the spring stiffnesses[5] of connectors of the 6-DOF-IPCM are chosen to be 19.58lb/in. The mechanism is assumed to be or near the unloaded position. Because the derivation of [K] is dependent on the coordinate system chosen, it is necessary to specify the coordinate system to which the stiffness matrix is referred. Let the coordinate system be denoted as Π. This coordinate system Π has its z-axis aligned with Ω. For the sake of convenience, the original point of the coordinate system Π is chosen right below the geometric center point c_0 of the bottom triangle 1.0625inch, which is a point with the coordinates $(x_{c2}, y_{c2}, z_{c2})=(2.2500, 1.2990, -1.0625)$ referred to the coordinate system Ω. This coordinate system is chosen to align with the coordinate system chosen for the experiment done in Chapter 3. The orientation of the coordinate system Π is different with Ω by a rotation of 47.5 degree about the z-axis of Ω. Thus, the stiffness matrix generated by using equation (2.43) for this kinematic model at or near its unloaded configuration, followed by a Euclidean coordinate transformation, is as follows:

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[5] The actual spring stiffness of each connector for the mechanism is slightly varied with each other. The variation is considered as the tolerance in the design which is provided by manufacturer with the manufacturer error within ±5%.

$$[K]_t = \begin{matrix} 23.3400 & 0.0000 & 0.0000 & 10.9285 & 84.2688 & -0.0006 \\ 0.0000 & 23.3400 & 0.0000 & -84.2688 & 10.9285 & 0.0007 \\ 0.0000 & 0.0000 & 70.7999 & 0.0018 & -0.0020 & -21.8569 \\ 10.9285 & -84.2688 & 0.0018 & 489.4321 & 0.0000 & -0.0032 \\ 84.2688 & 10.9285 & -0.0020 & 0.0000 & 489.4321 & -0.0012 \\ -0.0006 & 0.0007 & -21.8569 & -0.0032 & -0.0012 & 202.2579 \end{matrix}$$

2.6 Eigenvalue Problem

2.6.1 Formulations

The derivation of a stiffness matrix has been presented. In the following sections, one shall introduce the system characteristics by formulating eigenvalue problems from the definitions of eigenscrews, twist- and wrench-compliant axes, force- and rotation-compliant axes. It is assumed here that the stiffness matrix (correlation form) is in the most general case, asymmetric with an upper-left 3x3 symmetric submatrix and in the most special case, fully symmetric. As state earlier, a stiffness mapping, $\boldsymbol{S}: \boldsymbol{D} \rightarrow \boldsymbol{W}$ is generally represented as

$$\delta \underline{w} = [K]\delta \underline{D}, \tag{2.45}$$

where $\delta \underline{D} \in \boldsymbol{D}$, and $\delta \underline{w} \in \boldsymbol{W}$, is the induced wrench corresponding to the twist. The mapping $[K]$ is defined as the correlation form of stiffness mapping (see also Sec. 2.2) since a twist in axis coordinates is mapped into a wrench in ray coordinates. Let's recall the correlation operator $[\Delta]$ as defined in Equation (2.12) and reproduce it here as ,

$$[\Delta] = \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}. \tag{2.46}$$

Using the correlation operator $[\Delta]$, the twist is expressed in ray coordinates,

$$\delta \underline{d} = [\Delta]\delta \underline{D}. \tag{2.47}$$

The operator $[\Delta]$ is an orthogonal involution with the properties, $[\Delta]=[\Delta]^T$,$[\Delta]=[\Delta]^{-1}$, $[\Delta][\Delta]=I$. It should be noted that the operator $[\Delta]$ has three positive and three negative eigenvalues (1 and -1) [6].

Combining (2.45), (2.47) and making use of the fact, $[\Delta][\Delta]=I$, gives

$$\delta \underline{w} = [K][\Delta]\delta \underline{d} = [k]\delta \underline{d} \tag{2.48}$$

where the wrench, $\delta \underline{w}$, and the twist, $\delta \underline{d}$, are both in ray coordinates, and $[k]=[K][\Delta]$, The mapping $[k]$ is called the collineation form of stiffness mapping since the multiplication of two correlations is a collineation. Intuitively, the mapping $[k]$ maps a twist from ray to ray coordinates in e(3), where e(3) is Lie algebra of the Euclidean group. The corresponding eigenvalue problem is given by

$$\eta \hat{s} = [k]\hat{s}. \tag{2.49}$$

Equation (2.49) yields six eigenvalues η_i and six eigenvectors $\hat{s}_i$. The eigenvalues are referred to as *eigenstiffnesses* with units(force/radian) which are the ratios of the intensities of the wrenches to the twists. The six eigenscrews are described by these six eigenvectors $\hat{s}_i$, which are in ray coordinates.

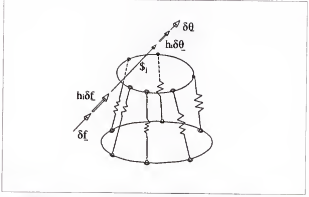

Figure 2.7 An eigenscrew $\$_i$ maps into itself in the space, i.e. the twist $\delta\underline{d}$ and the wrench $\delta\underline{w}$ are along the same screw $\$_i$ under the mapping [k].

Equation (2.49) states that a twist applied on an eigenscrew of the stiffness matrix induces a wrench along the same eigenscrew. Figure 2.7 illustrates an eigenscrew $\$_i$ which defines a twist with an angle $\delta\underline{\theta}$ is transformed into a wrench with an intensity δf along the same screw $\$_i$ under the stiffness mapping. These six eigenscrews span a completed six-dimensional vector space, since [k] is non-singular. It is implied that any wrench allowed by the stiffness matrix can be written as a linear combination of these six eigenscrews.

An equivalent form of the stiffness matrix eigenvalue problem can be expressed in axis coordinates using

$$[\Delta]\hat{S} = \hat{s}, \quad [k] = [K][\Delta]. \tag{2.50}$$

Substituting (2.50) into(2.49) yields

$$\eta[\Delta]\hat{S}=[K][\Delta][\Delta]\hat{S}. \tag{2.51}$$

Applying the fact [Δ][Δ]=I gives

$$\eta\hat{S}=[\Delta][K]\hat{S}. \tag{2.52}$$

In (2.52), the eigenscrew is in axis coordinates and is equivalent to (2.49) which is in ray coordinates. It is not difficult to deduce that both (2.49) and (2.52) generate the same eigenvalues and eigenscrews (see also Reference 19).

The inverse of stiffness is called compliance for a compliant system. They are related by,

$$[C]=[K]^{-1} \tag{2.53}$$

where [C] is generally a asymmetric 6x6 compliance matrix since the stiffness matrix [K] is asymmetric. Equation (2.45) can be expressed in terms of compliance,

$$\delta\underline{D}=[C]\delta\underline{w}, \tag{2.54}$$

where [C] is the correlation form of compliance mapping. Equation (2.54) states that a relative wrench between two rigid bodies is mapped into the change in twist that interacts between two rigid bodies by a compliance mapping.

Expressing the twist $\delta\underline{D}$ in ray coordinates by using (2.47) yields

$$\begin{aligned}\delta\underline{d} &=[\Delta][C]\delta\underline{w}\\ &=[c]\delta\underline{w},\end{aligned} \tag{2.55}$$

where [c]=[Δ][C], [Δ] is incorporated with [C] to produce [c]. Then (2.55) is the collineation form of compliance mapping, since [c] maps a twist in ray coordinates into a wrench in ray coordinates. Let $\hat{y}$ be denoted as the homogeneous coordinates of a screw that is mapped into itself by [c], then the eigenvalue problem of compliance mapping is given by,

$$\zeta\hat{y} =[c]\hat{y}. \tag{2.56}$$

where ζ is a scalar multiplier. Inverting [c] and dividing both sides of (2.56) by ζ yields

$$\frac{1}{\zeta}\hat{y}=[k]\hat{y}=[K][\Delta]\hat{y}. \tag{2.57}$$

Comparing (2.57) with (2.49) gives $(1/\zeta)=\eta$ and $\hat{y}=\hat{s}$. Hence the stiffness and compliance eigenvalue problems can be considered equivalent.

2.6.2 Eigen-Properties

For designing a 6-DOF-IPCM, the stiffness characteristics of this compliant system can be modeled by specifying the geometric parameters and the spring constants of its six connectors at its unloaded configuration. We desire to start from a stable configuration and have the infinitesimal motions of the compliant system consecutive around the neighborhood of the unloaded position of the system. There is at least one unloaded position where the platform is completely stable. Generally, the workspace of the system should be away from a singular position to maximize the stability in our application (see also Refs.31, 32, 33 and 34). A singular position occurs when the determinant of the Jacobian matrix is zero, or the determinant of the stiffness matrix is zero. Also, the properties of the stiffness mapping are an important information for designers.

A stiffness mapping is represented by a 6x6 matrix whose elements are all real. The mapping transforms real twists into real wrenches (and imaginary onto imaginary[7] if it has any). Here, the discussion is limited to the real cases. It is known that the stiffness matrix is symmetric and positive-definite when a compliant system is in its unloaded and stable position. The matrix becomes asymmetric when the system is subjected to an external resultant [3]. Here, it is recognized that there is a difference between correlation form, [K], and collineation form, [k], for the mapping. The followings are the summaries of properties of a stiffness matrix [K] when it is symmetric and positive-definite (see also Refs. 1 and 19).

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[7] Imaginary screws are induced by imaginary eigenstiffnesses. Imaginary eigenstiffnesses appear in pairs, i.e. they are conjugate. The induced imaginary eigenscrews are also conjugate. A screw is considered as a point in $\mathcal{P}_5$. Therefore, the imaginary screw may be considered as an imaginary point in $\mathcal{P}_5$, and conjugate imaginary screws are simply a pair of conjugate points in $\mathcal{P}_5$.

i) The signature of the matrix [K] is invariant under Euclidean transformations, but its eigenvalues change as the transformation occurs. The signature of [K] is defined as (π, ν, ς) which is the number of positive (π), negative (ν), and zero (ς) eigenvalues .

ii) The matrix [k], which is a collineation form, has a full set of eigenscrews that are all real. All of the eigenscrews have non-zero, finite pitches.

iii) [k] has three positive and three negative eigenvalues.

iv) The eigenscrews of [k] are real and co-reciprocal (This is not true when the siffness matrix is asymmetric). A set of screws are said co-reciprocal if the reciprocal product of every two of them is zero. However, they are not mutually perpendicular.

v) For each eigenscrew of the matrix [k], the corresponding eigenvalues and eigenscrew pitches must have the same sign.

vi) The eigenstiffnesses and eigenscrews of the [k] are invariant under Euclidean transformations. Of course, the coordinates of an eigenscrew change as the coordinate frame changes. However, the eigenscrew remains the same.

2.6.3 Classification of Stiffness Matrices

It is assumed that an eigenvalue η_1, for example, of a stiffness matrix [k] is a multiple root with multiplicity ε. If the non-trivial solution of eigenvalue problem defined in (2.49) possesses the factor $(\eta-\eta_1)^\varepsilon$, then for η_1 there corresponds ε linearly independent eigenscrews, since [k] is non-singular.

A classification of stiffness matrices can be performed using the number of multiplicity of eigenvalues and the corresponding eigenscrew systems.

Class 1, The eigenvalues, η_i, $i=1,2,...,6$, of a stiffness matrix [k] are all distinct. There are exactly six eigenscrews $\$_i$, which are invariant under the mapping. Any screw other than the six eigenscrews is mapped onto a different screw. Those six eigenscrews span a completed six dimensional vector space. Any two of these six eigenscrews form a two-system of screws. The general ruled surface on which the screws of the two-system lie is the cylindroid. Figure 2.8 illustrates the cylindroid where the midway section of the surface is in the *xy-plane*. The Euclidean coordinate system has *x*- ,*y*- and *z*- axes which are mutually perpendicular. Two given screws $\$_\alpha$, $\$_\beta$ having pitches h_α and h_β are directed along the *x*- and *y*- axes. These screws are called the principal screws of the surface.

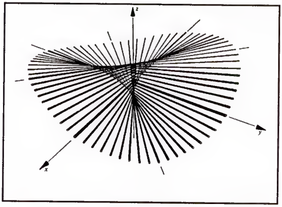

Figure 2.8 A general two-system of screws determines a cylindroid in the space. [35]

Class 2-1, The eigenvalue η_1 has multiplicity, for example, $\varepsilon_1=2$, and it is the only multiple root for the eigenvalue problem of [k]. The corresponding eigenscrews are not uniquely determined, and they pass through a common point A in a plane π. Any screw through the point A in plane π can be selected as an eigenscrew of the stiffness mapping [k]. The eigenscrews are a special two-system where the cylindroid collapses into a planar pencil of screws which have pitches of the same sign (Figure 2.9). The pair of eigenscrews can be determined by a method similar to the Gram-Schmidt orthogonalization procedure, which is discussed by Sugimoto and Duffy [36]. The method is to define a base of a screw system *V* that is a set of mutually orthogonal screws. When an eigenscrew is selected, the other one is constructed by satisfying orthogonality. Let the

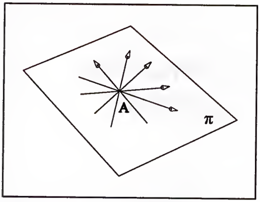

Figure 2.9 A planar pencil of eigenscrews is a special two-system of screws, which is the cylindroid collapsing onto a plane.

method be called Sugimoto-Duffy orthogonalization screws.

The remaining four eigenscrews of the matrix [k] are uniquely determined. Therefore, a planar pencil of eigenscrews plus four unique eigenscrews are invariant under this mapping.

Class 2-2, There are two eigenvalues, η_1 and η_2, with multiplicity, $\varepsilon_1 = \varepsilon_2 = 2$. This stiffness matrix can contain no more than two double roots. The corresponding eigenscrews for each eigenvalue form a planar pencil of screws (Figure 2.10). The remaining two eigenscrews are uniquely determined. In this case, two planar pencils of screws plus two unique screws are invariant under this mapping.

Class 3-1, The eigenvalue, η_1, has multiplicity, $\varepsilon_1 = 3$. There is a bundle of eigenscrews through a point O for the matrix [k] (Figure 2.11). Every screw through a point O is an eigenscrew. The remaining three distinct eigenvalues correspond to three unique eigenscrews that form a general three-system. Thus, a bundle of screws plus three uniquely determined screws are invariant under this mapping.

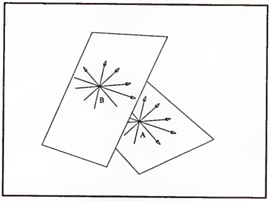

Figure 2.10 Two planar pencils of screws.

Class 3-1-2, There are two eigenvalues, η_1 and η_2, with multiplicity, $\varepsilon_1 = 3$ and $\varepsilon_2 = 2$. The system has a bundle of eigenscrews corresponding to eigenvalue η_1 and a planar pencil of eigenscrews corresponding to eigenvalue η_2. The remaining eigenscrew is uniquely defined in space.

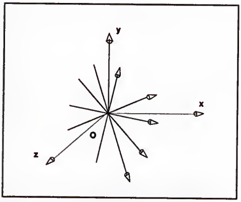

Figure 2.11 A bundle of screws in the space.

Class 3-2, There are two eigenvalues, η_1 and η_2, with multiplicity, $\varepsilon_1=\varepsilon_2=3$. This is the only possible case for the matrix [k] having two roots with multiplicity 3. Each eigenvalue corresponds to a bundle of eigenscrews. Thus, the stiffness matrix in this case has two bundles of screws which are invariant under the mapping.

An eigenvalue with multiplicity greater than 3 cannot occur for the mapping, since the matrix [k] has three positive and three negative eigenvalues.

2.7 Twist- and Wrench-Compliant Axes

The twist-compliant axes and wrench-compliant axes are other important characteristics for the mapping. A twist applied to the corresponding twist-compliant axis (TCA) produces a pure couple in the direction of the axis. A wrench applied to the corresponding wrench-compliant axis (WCA) produces a pure linear displacement in the direction of the axis. Three twist- and wrench-compliant axes can be determined by solving each corresponding eigenvalue problem for a non-singular stiffness matrix. The twist- and wrench-compliant axes posses many important properties that are detailed in the following sections.

2.7.1 Twist-Compliant Axes

The twist-compliant axes of a stiffness matrix are found by using the stiffness relationship $\delta\underline{w}=[K][\Delta]\delta\underline{d}$. A twist and a pure couple in the direction of the twist are related by,

$$\eta_\theta \delta \underline{w}_\theta = [K][\Delta]\delta \underline{d}_\theta, \qquad (2.58)$$

where $\delta\underline{d}_\theta=[\delta\underline{\theta};\ \delta\underline{x}]$, and $\delta\underline{w}_\theta = [\underline{0};\delta\underline{\theta}]$; $\delta\underline{\theta}$ is the twist direction, and η_θ is the ratio of intensity of the couple to the angular displacement magnitude, an angular stiffness. Rewriting (2.58) as an eigenvalue problem gives

$$\eta_\theta[\Delta][\Gamma]\delta\underline{d}_\theta = [K][\Delta]\delta\underline{d}_\theta \qquad (2.59)$$

where

$$[\Gamma] = \begin{pmatrix} I_3 & 0_3 \\ 0_3 & 0_3 \end{pmatrix}. \qquad (2.60)$$

(2.59) is pre-multiplied by$([K][\Delta])^{-1}$ and divided by η_θ yields,

$$\zeta_\theta\ \delta\underline{d}_\theta = [\Delta][K]^{-1}[\Delta][\Gamma]\delta\underline{d}_\theta, \qquad (2.61)$$

where $\varsigma_\theta=1/\eta_\theta$. The compliance ς_θ is the inverse of stiffness, η_θ. Now, we denote $[K]^{-1} = [C]$, and let

$$[C] = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \qquad (2.62)$$

where C_{11}, C_{12}, C_{21}, and C_{22} are all 3x3 matrices. These four sub-matrices of [C] may not be symmetric. Substituting (2.60) ,(2.62) and $[\Delta]$, into (2.61) and simplifying (2.62) yields

$$\zeta_\theta\delta\underline{d}_\theta = [C_\theta]\delta\underline{d}_\theta, \qquad (2.63)$$

where the substitution

$$[C_\theta] = \begin{pmatrix} C_{22} & 0 \\ C_{12} & 0 \end{pmatrix} \tag{2.64}$$

has been introduced. Therefore, the eigenvalue problem of twist-compliant axes is reduced to solve the eigenvalue problem of a 6x6 matrix $[C_\theta]$ with rank three. There are three zero eigenvalues, and the three remaining non-zero eigenvalues corresponding to three eigenvectors, which are the homogeneous coordinates of three eigenscrews. The three eigenscrews are the twist-compliant axes. Further, from (2.63),

$$\zeta_\theta \begin{pmatrix} \delta\underline{\theta} \\ \delta\underline{x} \end{pmatrix} = \begin{pmatrix} C_{22} & 0 \\ C_{12} & 0 \end{pmatrix} \begin{pmatrix} \delta\underline{\theta} \\ \delta\underline{x} \end{pmatrix} \tag{2.65}$$

which can be reduced to the form

$$\zeta_\theta \begin{pmatrix} \delta\underline{\theta} \\ \delta\underline{x} \end{pmatrix} = \begin{pmatrix} C_{22}\delta\underline{\theta} \\ C_{12}\delta\underline{\theta} \end{pmatrix}. \tag{2.66}$$

Examining only upper part of (2.66) obtains

$$\varsigma_\theta \delta\underline{\theta} = C_{22}\delta\underline{\theta}. \tag{2.67}$$

The directions of twist-compliant axes are the same as the eigenvectors of the matrix C_{22}.

The three eigenscrews are linearly independent and form a three-system of screws in e(3) (see Ref. 35 for detail about general three-system). When the 6-DOF-IPCM is unloaded, the stiffness matrix [K] is symmetric, while C_{22} and C_{12} are thus both symmetric. The eigenvalue problem of matrix C_θ thus always has three real and non-zero eigenvalues corresponding to three real eigenscrews. The three eigenscrews are mutually [I]-orthogonal (i.e., perpendicular), where [I] is a 6x6 identity matrix. They are not necessarily to be mutually co-reciprocal (i.e. [Δ]-orthogonal). However, two twist-compliant axes are reciprocal, if $\delta\underline{\theta}_1^T C_{22}\, \delta\underline{x}_2 - \delta\theta_2^T C_{22}\, \delta x_1 = 0$.

The eigen-property of perpendicularity does not exist when the system deflects far form its unloaded configuration.

If the characteristic polynomial of $[C_\theta]$ has multiple root with multiplicity r, and $r \leq 3$, since it is rank three. The corresponding eigenscrews can be constructed as orthogonal screws by using Sugimoto-Duffy orthogonalization screws.

2.7.2 Wrench-Compliant Axes

The wrench-compliant axis of a stiffness matrix can be formulated from definition. A wrench and pure linear displacement in the direction of the wrench are related by,

$$\alpha_f\, \delta\underline{w}_f = [K][\Delta]\delta\underline{d}_f, \tag{2.68}$$

where $\delta\underline{d}_f = [\underline{0}; \delta\underline{f}]$, and $\delta\underline{w}_f = [\delta\underline{f}; \delta\underline{m}_0]$; $\delta\underline{f}$ is the direction of the wrench. Rewriting (2.68) as an eigenvalue problem yields,

$$\alpha_f \delta\underline{w}_f = [K][\Delta][\Delta][\Gamma]\delta\underline{w}_f, \tag{2.69}$$

where α_f is the ratio of intensity of force magnitude to the linear displacement magnitude, a linear stiffness. Since $[\Delta][\Delta]=I$, therefore,

$$\alpha_f \delta\underline{w}_f = [K][\Gamma]\delta\underline{w}_f \tag{2.70}$$

Equation (2.70) is the eigenvalue problem with six eigenvalues α_{fi} and six eigenscrews w_{fi}. Let [K] be subdivided into four 3x3 sub-matrices and denoted as,

$$[K] = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}. \qquad (2.71)$$

The matrix multiplication [K][Γ] in (2.70) is simplified as

$$[K][\Gamma] = \begin{pmatrix} K_{11} & 0 \\ K_{21} & 0 \end{pmatrix} = [K_f]. \qquad (2.72)$$

Hence, the eigenvalue problem of wrench-compliant axes becomes

$$\alpha_f \delta \underline{w}_f = [K_f] \delta \underline{w}_f \qquad (2.73)$$

$[K_f]$ is a matrix of rank three and thus has three zero eigenvalues and the three remaining non-zero eigenvalues corresponding to three eigenvectors, which are the homogeneous coordinates of three eigenscrews. These eigenscrews are defined as the wrench-compliant axes.

Further, (2.73) can be reduced to,

$$\alpha_f \begin{pmatrix} \delta \underline{f} \\ \delta \underline{m}_0 \end{pmatrix} = \begin{pmatrix} K_{11} \delta \underline{f} \\ K_{21} \delta \underline{m}_0 \end{pmatrix}. \qquad (2.74)$$

Examining only upper part of (2.74) obtains

$$\alpha_f \delta \underline{f} = K_{11} \delta \underline{f}. \qquad (2.75)$$

Since the stiffness matrix [K] is generally asymmetric with its upper-left 3x3 sub-matrix K_{11} symmetric, equation (2.73) has three real and non-zero eigenvalues that are corresponding to three real eigenvectors, which are the homogeneous coordinates of the eigenscrews of $[K_f]$. The directions of the wrench-compliant axes are the same as the eigenvectors of the matrix K_{11}. The three eigenscrews form a three-system in e(3). They are always [I]-orthogonal, i.e., mutually perpendicular to each other no matter whether the system is unloaded or loaded. The two wrench-compliant axes, $\delta\underline{w}_{f1}$ and $\delta\underline{w}_{f2}$, respectively, are reciprocal, if $\delta\underline{m}_{01} K_{11} \delta\underline{f}_2 - \delta\underline{m}_{02} K_{11} \delta\underline{f}_1 = 0$ and K_{21} are symmetric.

If the characteristic polynomial of K_f has multiple roots with multiplicity ε, where $\varepsilon \leq 3$, since $[K_f]$ is of rank three. The corresponding eigenscrews can be constructed as orthogonal screws by using Sugimoto-Duffy orthogonalization screws.

2.7.3 Remarks on TCA and WCA

The twist- and wrench-compliant axes form two reciprocal three-systems in space if and only if the corresponding induced translation and couple are reciprocal, and stiffness matrix [K] is symmetric. This property can be understood from the point of view of projective geometry. The geometric meaning of this reciprocal three-systems screws is the incidence of a point in a hyperplane. The image of the point and hyperplane under the mapping preserves the incidence that is the hyperplane passing through the point.

For simplification, we consider a linear transformation of real analytic projective geometry on a plane. A correlation of S_2 may be expressed in the form

$$b = [Q]U, \tag{2.76}$$

where [Q] is a 3x3 transformation matrix, U is a 3x1 column vector of the coordinates of a point, and b is 3x1 column vector of the coefficient of a line equation. The correlation [Q] maps each point U into a line b containing point V such that

$$V^T b = 0. \tag{2.77}$$

Equation (2.77) is the condition that the point V should be incident on the line b corresponding to the point U under the correlation (2.76). Similarly, if each point V on the line b is transformed onto a line g containing point Y under the correlation (2.76), then,

$$g=[Q]V \ and \ g^{T}Y=0. \tag{2.78}$$

If U is incident on the line g then

$$g^{T}U=0. \tag{2.79}$$

Inverting (2.76) and substituting the result into (2.78) yields

$$g^{T}[Q]^{-1}b=0. \tag{2.80}$$

From (2.78), substituting g=[Q]V into (2.80) yields

$$V^{T}[Q]^{T}[Q]^{-1}b=0. \tag{2.81}$$

Comparing (2.81) with (2.77) concludes $[Q]^T[Q]^{-1}=I$, i.e., the matrix [Q] is symmetric. The mapping [Q] is a polarity.

Under a polarity, the line b is called the polar of the point U and the point is called the pole of the line. It is similar to the relationship between V and g. From the above demonstration, a point U is on the polar of V if and only if V is on the polar of U. There is a simple way to construct the polar of a point with a conic defined by [Q]. Suppose U is an exterior point of the conic, then the corresponding polar of the point U is b, which is obtained by connecting two tangent points on the conic. Any point B on the line b, i.e. $b^TB=0$, has the corresponding polar u, which is obtained by connecting B to the two tangent points on the conic. Then

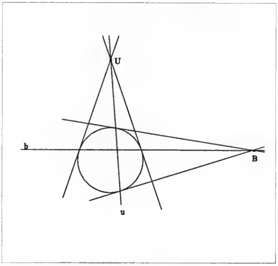

Figure 2.12 Point B and its corresponding polar u with respect to a conic.

the polar u of the point B must pass through the point U, which satisfies equation (2.77), i.e., $u^T T=0$ (Figure 2.12).

A screw is considered as a point in P^5. If two screws (one in ray coordinates and another in axis coordinates) satisfy the equation (2.77), the two screws are reciprocal. The correlation form of a stiffness mapping transforms an infinitesimal wrench, $\underline{w}_f$, in ray coordinates into a pure translation, $\underline{D}_f$, in axis coordinates by stiffness matrix [K],

$$\underline{w}_f=[K]\underline{D}_f\ . \tag{2.82}$$

The polar of a point $\underline{D}_\theta$ on the line $\underline{w}_\theta$ under the same mapping is represented as

$$\underline{w}_\theta=[K]\underline{D}_\theta, \tag{2.83}$$

which states that an infinitesimal twist $\underline{D}_\theta$ in axis coordinates is mapped into a pure couple $\underline{w}_\theta$. The pure couple $\underline{w}_\theta$ is always reciprocal to pure translation $\underline{D}_f$, because they satisfy the relationship,

$$\underline{w}_\theta{}^T\underline{D}_f = 0. \tag{2.84}$$

This is an incidence relationship and states also that a point, $\underline{D}_f$, lies on a line, $\underline{w}_\theta$. Substituting (2.82),(2.83) into (2.84) yields

$$D_\theta{}^T[K]^T[K]^{-1}w_f = 0. \tag{2.85}$$

if [K] is symmetric, the equation resolves to,

$$D_\theta{}^T w_f = 0, \tag{2.86}$$

where the twist-compliant axis is reciprocal to the wrench-compliant axis. The statement is proven completely.

2.7.4 Principal Screws of TCA and WCA

For a general three-system, the principal screws have the properties of reciprocity, perpendicularity and intersection at a common point. Additionally, the principal screws have stationary values of pitch [21]. From above definitions, it is found that the principal screws of wrench- (or twist-) complaint axes only belong to the categories of general three-systems and the first two special three-systems as Hunt's classification (see Ref.35 for Hunt's classification of screw systems.).

Suppose that the wrench-compliant axes have three principal screws α ,β and γ with pitches h_α, h_β and h_γ, respectively, where h_α is the maximum pitch and h_γ is the minimum pitch, and they contain no pitch of screw which is greater than the largest or less than the least of the pitches of principal screws. According to Hunt's classification of screws, it is found that the possible cases of principal screws are:

(1) general three-system: $h_\alpha \neq h_\beta \neq h_\gamma$, all finite.

(2) The first special-system: $h_\alpha = h_\beta$, h_γ, all finite.

(3) The second special-system: $h_\alpha = h_\beta = h_\gamma$, all finite.

(4) None of the wrench- (or twist-) compliant axes are of pitch infinity.

In general, the common points of the corresponding principal screws of three wrench-compliant axes and three twist-compliant axes are not concurrent. However, they are concurrent if the stiffness matrix is symmetric.

Therefore, for two reciprocal three-systems, their principal screws have the properties of a general three-system and form a coincident trihedral. Each pair of collinear principal screws along each trihedral axis has pitches equal in magnitude and opposite in sign. Here, if the collinear principal screws have zero pitch, then they are rotation- and force-compliant axes. The pair defines a compliant axis that passes through the center of elasticity which is proposed by Lipkin and Patterson [19,21]. If all of the principal screws have pitch zero then three pairs of collinear principal screws define completely a compliant three-system. The point of the intersection is the center of compliance. Every axis through this point is a compliant axis. The center of the reciprocal three-systems is defined as the center of elasticity. It is clear that the center of compliance is a special case of the center of elasticity.

2.8 Force- and Rotation-Compliant Axes

The force- and rotation-compliant axes are the special case of twist- and wrench-compliant axes with zero pitch. A force applied to the force-compliant axis induces a translation in the direction of the axis. A rotation applied about the rotation-compliant axis induces a couple in the direction of the axis. Rotation- and force-compliant axes form a two-system or determine a cylindroid.

Analytically, these relationships of the force with the induced translation, and rotation with the induced couple are expressed in terms of stiffness as

$$\eta_1 \begin{pmatrix} \underline{f} \\ \underline{r} X \underline{f} \end{pmatrix} = [K][\Delta] \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix} \tag{2.87}$$

and

$$\eta_2 \begin{pmatrix} \underline{0} \\ \underline{\theta} \end{pmatrix} = [K][\Delta] \begin{pmatrix} \underline{\theta} \\ \underline{r} x \underline{\theta} \end{pmatrix}, \tag{2.88}$$

where $\underline{f}$ is a unit vector along the direction of force, and $\underline{\theta}$ is a unit vector along the direction of rotation. The scalar multipliers η_1 and η_2 are linear and rotational stiffnesses.

A wrench is the linear combination of a force and the couple in the direction of the force axes. A twist is the linear combination of a rotation and the translation in the direction of rotational axes. The concepts of the force- and rotation-compliant axes provide a foundation to express the motion of rigid bodies and define a compliant axis.

An axis is a compliant axis if an applied rotation produces a pure couple in the axis, and an applied force produces a pure translation in the axis. From the collineation form of stiffness mapping, a compliant axis exists if and only if there are two collinear eigenscrews that have eigenpitches of equal magnitude and opposite sign. The pair of collinear eigenscrews are also reciprocal.

It can be proved by assuming that there exists a compliant axis, $\underline{a}$. Let the origin be located on the compliant axis, so $\underline{r}=0$. From the definition of compliant axis, it has

$$\eta_1\begin{pmatrix}\underline{a}\\ \underline{0}\end{pmatrix}=[K][\Delta]\begin{pmatrix}\underline{0}\\ \underline{a}\end{pmatrix}, \tag{2.89}$$

$$\eta_2\begin{pmatrix}\underline{0}\\ \underline{a}\end{pmatrix}=[K][\Delta]\begin{pmatrix}\underline{a}\\ \underline{0}\end{pmatrix} \tag{2.90}$$

where $\underline{a}$ is a unit vector in the direction of the compliant axis, $\hat{a}$. Suppose that two other screws, $\$_1$ and $\$_2$, respectively, along the compliant axis have the forms,

$$\$_1=\begin{pmatrix}\underline{a}\\ h_1\underline{a}\end{pmatrix}, and\ \$_2=\begin{pmatrix}\underline{a}\\ h_2\underline{a}\end{pmatrix}, \tag{2.91}$$

where $h_1 \neq h_2$, and h_1, $h_2 \neq 0$, and they are arbitrary. The average of the two screws in (2.91) yields,

$$\frac{\$_1+\$_2}{2}=\begin{pmatrix}\underline{a}\\ \frac{(h_1+h_2)\underline{a}}{2}\end{pmatrix} \tag{2.92}$$

The difference of the two screws in (2.91) is divided by (h_1-h_2), having the form,

$$\frac{\$_1-\$_2}{(h_1-h_2)}=\begin{pmatrix}\underline{0}\\ \underline{a}\end{pmatrix}. \tag{2.93}$$

The equation (2.92) can be rewritten as,

$$\frac{\$_1+\$_2}{2}=\begin{pmatrix}\underline{a}\\ \underline{0}\end{pmatrix}+\frac{(h_1+h_2)}{2}\begin{pmatrix}\underline{0}\\ \underline{a}\end{pmatrix}. \tag{2.94}$$

Applying (2.93) into (2.94) and regrouping the terms yields

$$\begin{pmatrix}\underline{a}\\ \underline{0}\end{pmatrix}=\frac{h_1\$_2+h_2\$_1}{h_1-h_2}. \tag{2.95}$$

Applying (2.93) and (2.94) into (2.89), (2.90) generates the forms,

$$\eta_1(h_1\$_2-h_2\$_1)=[K][\Delta](\$_1-\$_2) \tag{2.96}$$

and

$$\eta_2(\$_1-\$_2)=[K][\Delta](h_1\$_2-h_2\$_1) \tag{2.97}$$

if (2.97) is divided by h_2 and added to (2.96) and regrouped, the related terms yields,

$$(\eta_1 h_1-\frac{\eta_2}{h_2})\$_2+(\frac{\eta_2}{h_2}-\eta_1 h_2)\$_1=[K][\Delta](\frac{h_1}{h_2}-1)\$_2 \tag{2.98}$$

Since h_1 and h_2 are arbitrary, the setting $\eta_2/h_2 - \eta_1 h_2 = 0$ yields

$$h_2^{\,2} = \frac{\eta_2}{\eta_1}. \tag{2.99}$$

Applying (2.99) to (2.98) and dividing both sides of (2.98) by $(h_1/h_2 - 1)$, (2.98) is reduced to,

$$\frac{h_2}{h_1 - h_2}\left(\eta_1 h_1 - \frac{\eta_2}{h_2}\right)\$_2 = [K][\Delta]\$_2 \tag{2.100}$$

or

$$\lambda_2 \$_2 = [K][\Delta]\$_2 \tag{2.101}$$

where the substitution

$$\lambda_2 = \frac{h_2}{h_1 - h_2}\left(\eta_1 h_1 - \frac{\eta_2}{h_2}\right) \tag{2.102}$$

has been introduced. Therefore, $\$_2$ is a eigenscrew and λ_2 is the eigenstiffness of $[K][\Delta]$ (i.e. $[k]$, a collineation form of stiffness mapping).

Similarly, applying $(2.97)/h_1 + (2.96)$ and regrouping the related terms yields,

$$\left(\eta_1 h_1 - \frac{\eta_2}{h_1}\right)\$_2 + \left(\frac{\eta_2}{h_1} - \eta_1 h_2\right)\$_1 = [K][\Delta]\left(1 - \frac{h_2}{h_1}\right)\$_1, \tag{2.103}$$

Setting $(\eta_1 h_1 - \eta_2/h_1) = 0$ yields

$$h_1^{\,2} = \frac{\eta_2}{\eta_1}. \tag{2.104}$$

Applying (2.104) to (2.103) and dividing both sides of (2.103) by $(1-h_2/h_1)$, (2.103) is reduced as,

$$\frac{h_2}{h_1-h_2}\left(\frac{\eta_2}{h_1}-\eta_1 h_2\right)\$_1=[K][\Delta]\$_1 \tag{2.105}$$

or

$$\lambda_1\$_1=[K][\Delta]\$_{21} \tag{2.106}$$

where the substitution

$$\lambda_1=\frac{h_2}{h_1-h_2}\left(\frac{\eta_2}{h_1}-\eta_1 h_2\right) \tag{2.107}$$

has been introduced. $\$_1$ is a eigenscrew and λ_1 is the eigenstiffness of [K][Δ].

Subtracting (2.99) from (2.104) gives,

$$h_1^2-h_2^2=0. \tag{2.108}$$

Since $h_1 \neq h_2$, it can be deduced that $h_1=-h_2$. Substituting the result in (2.102) and (2.107) yields the conclusion that $\lambda_1=-\lambda_2$. The proof shows that a compliant axis of [K][Δ] ([k]=[K][Δ], a collineation form of stiffness mapping) is defined by two collinear eigenscrews having eigenpitches with the same magnitude but of opposite sign. It also shows that the eigenstiffnesses have the same magnitude and opposite sign.

The demonstration shows that a compliant axis of a stiffness mapping exists if and only if the collineation form of the stiffness mapping has two collinear eigenscrews with their eigenpitches having the same magnitude and opposite sign. The result also provides that a compliant system can have compliant axis at any position as long as the condition is satisfied.

It is known that the eigenscrews of [k] are co-reciprocal if the stiffness matrix [k] is symmetric. If the compliant system has a compliant axis and [k] is symmetric, the remaining four eigenscrews intersect the axis of the collinear pair, which define a compliant axis, at right angles, but those eigenscrews do not necessarily meet at a common point [21]. If the eigenscrews are not co-reciprocal, then the compliant axis and the other four eigenscrews are skew.

If the stiffness matrix is symmetric and there exists two pairs of collinear eigenscrews, then the remaining two eigenscrews form a third collinear pair and these collinear pairs meet at a common point. The common point of the intersection is the center of compliance for the given 6-DOF-IPCM which has been defined in Reference 21.

If the stiffness matrix is not symmetric, the two pairs of collinear eigenscrews and the remaining eigenscrews are skew. It should be understood that the property of reciprocity does not hold when the system is under large load.

CHAPTER 3
EXPERIMENTAL INVESTIGATION OF STIFFNESS MAPPINGS

3.1 Theory

An experimental investigation of stiffness mappings is divided into two parts: one is an experimental measurement of twist and wrench matrices and the determination of stiffness matrices, the other is to apply the theoretically modeled stiffness matrix and an experimental stiffness matrix to control the force and position for a robot manipulator to examine the stiffness model wether it satisfies the desired functions. The latter will be detailed in Sections 3.5 and 3.6. In this section, and the following two sections will present the theory, instruments and procedure for this experiment.

As defined in Chapter 2, a stiffness mapping, $\boldsymbol{S}: \boldsymbol{D} \rightarrow \boldsymbol{W}$, is represented by a 6x6 stiffness matrix [K]. The stiffness mapping transforms a small twist into a wrench for a 6-DOF-IPCM. Suppose that $\delta\underline{D}$ is the axial coordinates of an infinitesimal twist and $\delta\underline{D} \in \boldsymbol{D}$, and $\delta\underline{w}$ is the ray coordinates of a finite wrench and $\delta\underline{w} \in \boldsymbol{W}$. Rewrite Equation (2.1),

$$\delta\underline{w} = [K]\delta\underline{D}. \tag{3.1}$$

This stiffness mapping is one-to-one and it maps a twist onto a corresponding wrench.

A determination of the stiffness matrix cannot be completed by taking a single wrench-and-twist pair as defined in equation (3.1). Six wrench-and-twist pairs are required because the stiffness matrix is a 6x6 matrix. The twist space is completely defined by a set of six linearly independent twist vectors. Any twist vector in the twist space under the mapping is a linear combination of these six twist vectors. The six wrench vectors which correspond to these six twist vectors are also linearly independent. The six wrench vectors define a basis for the wrench space. Any wrench vector in this wrench space under the mapping is a linear combination of these six wrench vectors. Once the twist space and the wrench space are completely defined, the stiffness matrix for the mapping can be specified.

To clarify this concept, suppose that $[\delta\underline{D}]$ is a 6x6 matrix which takes six linearly independent twists, $\delta\underline{D}_1$, $\delta\underline{D}_2$, $\delta\underline{D}_3$, $\delta\underline{D}_4$, $\delta\underline{D}_5$ and $\delta\underline{D}_6$ in $\boldsymbol{D}$, as its column vectors, such that $[\delta\underline{D}]= (\delta\underline{D}_1\ \delta\underline{D}_2\ \delta\underline{D}_3\ \delta\underline{D}_4\ \delta\underline{D}_5\ \delta\underline{D}_6)$. Correspondingly, $[\delta\underline{w}]$ is a 6x6 matrix which takes a set of six linearly independent wrenches, $\delta\underline{w}_1$, $\delta\underline{w}_2$, $\delta\underline{w}_3$, $\delta\underline{w}_4$, $\delta\underline{w}_5$ and $\delta\underline{w}_6$ in $\boldsymbol{W}$, as its column vectors, such that $[\delta\underline{w}]= (\delta\underline{w}_1\ \delta\underline{w}_2\ \delta\underline{w}_3\ \delta\underline{w}_4\ \delta\underline{w}_5\ \delta\underline{w}_6)$. For each twist $\delta\underline{D}_i$ and wrench $\delta\underline{w}_i$, they are related by the formula $\delta\underline{w}_i=[K]\delta\underline{D}_i$, where i=1,2,3..,6. Hence,

$$[\delta\underline{w}]=[K][\delta\underline{D}]. \tag{3.2}$$

It should be noted that equation (3.2) is valid when the twists and wrenches are all small.

From Equation (3.2), there are three solutions for each one of the variables, $[\delta\underline{D}]$, $[\delta\underline{w}]$, and $[K]$:

(i) If $[\delta\underline{D}]$ and $[K]$ are known, it is required to determine $[\delta\underline{w}]$.

In this case, if one supplies a set of six small twists, which are linearly independent, and the stiffness matrix is known, then the set of six corresponding wrenches, which are linearly independent, can be computed from Equation (3.2).

(ii) If $[\delta\underline{w}]$ and $[K]$ are known, it is required to determine $[\delta\underline{D}]$.

If a wrench matrix is given and the stiffness matrix has been known, then the twist matrix can be obtained by inverting $[K]$ in equation (3.2) so that

$$[\delta \underline{D}] = [K]^{-1}[\delta \underline{w}]. \tag{3.3}$$

(iii) If $[\delta \underline{w}]$ and $[\delta \underline{D}]$ are given, it is required to determine [K].

The stiffness matrix can be found by inverting the twist matrix $[\delta \underline{D}]$ in Equation (3.2) to give

$$[K] = [\delta \underline{w}][\delta \underline{D}]^{-1}. \tag{3.4}$$

From the above definitions, the experimental investigation of a stiffness mapping could be accomplished by taking any of equations (3.2), (3.3), and (3.4) for a desired application.

When the stiffness matrix is unknown, there are no means to directly measure [K] by experiment. It is necessary to measure a pair of twist and wrench matrices and to compute the [K] by using equation (3.4). The instruments and procedure for the experimental measurement of twist and wrench matrices will be described in the following sections.

3.2 Instruments

The following instruments are used in the experiment to generate twist and wrench matrices for a 6-DOF-IPCM at Center of Intelligent Machines and Robotics (CIMAR) Lab, Department of Mechanical Engineering, University of Florida.

6-DOF in-parallel compliant mechanism (6-DOF-IPCM)

A prototype 6-DOF-IPCM based upon the conceptual design and the specified model shown in Sections 2.4 and 2.5 is constructed in this study. Figure 3.1 illustrates a

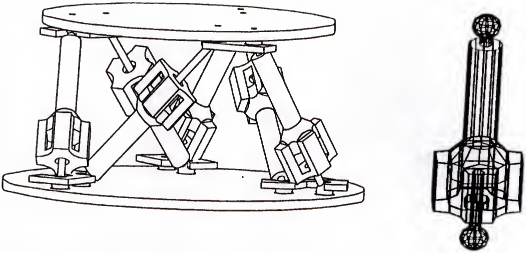

Figure 3.1 A conceptual drawing for a 6-DOF in-parallel compliant mechanism and one of its connectors.

drawing of the 6-DOF-IPCM and one of its connectors. The 6-DOF-IPCM has six connectors to connect two rigid plates. Thus, it has six degrees of freedom, three linear and three angular. The movements of the platform are relative to each other. If one of the platforms is defined as *the base*, the other, which is defined as *the top movable platform*, can perform a motion relative to the base.

The connector is comprised of two ball-and-socket joints, a small slender rod, and a housing. The slender rod with one side connected to a spherical ball is like a part of the ball-and-socket joint; the other side is free to slide along the axial direction of the connector. The rod is guided by a pair of "RULON" nonmetallic bearings (manufactured by Dixon Industries Corp.)[1] to reduce the friction and wearing during motion. The slender rod slides inside of the housing to allow only translation along the axis of the connector. When an external force is applied to the mechanism, the rod reacts to the external wrench to push against a spring system and to move along the housing to produce a deflection on the springs, i.e. a linear displacement along the axis of the housing. The slender rod passes through the center of a round disk which is fixed to the rod to separate a pair of compression springs. These compression springs are equally preloaded. This arrangement is as shown in Figure 3.2 and makes the two compression springs to be able to simultaneously react to the applied force and against each other.

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[1] Distributed by Small parts Inc.,Miami Lakes, FL, Catalog No. 14.

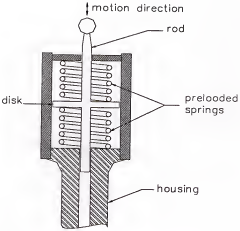

Figure 3.2 A connector of a 6-DOF-IPCM

Figure 3.3 shows one of the connectors of the prototype 6-DOF-IPCM. Figure 3.4 shows the assembly of the prototype mechanism which has six linearly independent connectors acting in parallel. The six connectors are arranged in pair-wise for avoiding mechanism interference.

Force/torque sensor

A Lord ATI Corp. Model 15/50 force/torque sensor was available in CIMAR Lab. This sensor was used to record the force/torque during the measurement. The detailed technique supports can be found in User's Manual [37].

GE-P60 Robot.

A GE-P60 robot is a serial-chain robot manipulator which consists of three main components: robot arms, teach pendant, and RC1560 controller. The general information and technical supports can be founded in GE-P60 User's Manual [38] and will not be detailed here.

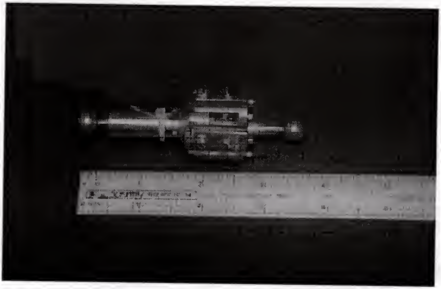

Figure 3.3 One of the connectors of a 6-DOF-in-parallel compliant mechanism.

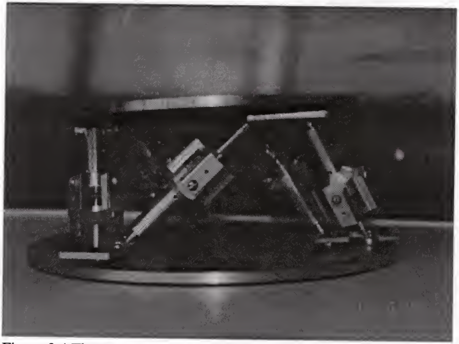

Figure 3.4 The assembly of the prototype 6-DOF in-parallel compliant mechanism is classified as a special 6-6 platform.

IBM Compatible 486-66 Personal Computer

The IBM Compatible 486-66 personal computer used in this project (called "commanding computer" or "CC") is the main unit to send commands to control the GE-P60 robot. In the commanding computer, there are three motion control cards [39] (Each of cards controls two GE-P60 robot joints) used to communicate with the main control unit (RC1560 controller) of the GE-P60 robot to instruct the GE-P60 robot for the desired motion. Each commanded motion is fed back to the commanding computer to monitor the position and orientation of the end-effector of the GE-P60 robot.

Arrangement

The end-effector of the robotic manipulator was a slim aluminum bar fixed to a large immovable table (considered as the ground). The motions are controlled by the IBM compatible 486-66 personal computer with three motion control cards.

The designed 6-DOF-IPCM is installed between the last link and the end-effector of the given robot manipulator. Thus the mechanism enables the last link of the robot manipulator to have six degrees of freedom and to be free from any constraints imposed by the end-effector. The force/torque sensor is located right at the end of the last link of the robot manipulator. These components constitute the apparatus in this sequence: robot series-links, force/ torque sensor, compliant mechanism, and end-effector which are as shown in Figure 3.5.

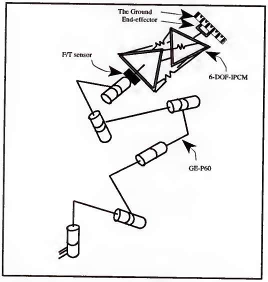

Figure 3.5 A general six-revolute-joints manipulator with a 6-DOF in-parallel mechanism.

3.3 Experiment Procedure

It is required to experimentally determine the stiffness mapping of a 6-DOF in-parallel compliant mechanism. In this section, the procedure for generating a twist matrix and a wrench matrix is detailed. Once the two matrices are obtained, the stiffness matrix is obtained by using equation (3.4).

In the measurement, it is supposed that the end-effector of the GE-P60 robot manipulator is fixed to the ground. Any force applied to the end-effector by the constraint is recorded by the force/torque sensor while the robot system changes its configuration. During the installation, it is desired that the mechanism remains at its unloaded position.

The stiffness matrices are coordinate-dependent. The coordinates of an infinitesimal motion caused by a change of the applied wrench depends upon the location of the reference frame. This means that the elements of a stiffness matrix must depend upon the location of the reference frame. Moreover, a pair of stiffness matrices in different reference frames are related by the congruence $[K'] = [E]^T[K][E]$, where [E] is a coordinate transformation matrix, which have been defined in Equation (2.5). In this case, the reference point can be chosen randomly based upon the consideration of calculation needed.

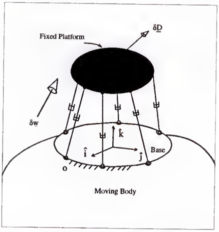

Figure 3.6 The coordinate frame for the motion of a platform.

It is practical to choose a coordinate system at a certain point of the bottom platform (or the base) of the 6-DOF-IPCM as shown in Figure 3.6. The reason is the force/torque sensor used to record the amount of static force loaded to a moving body is mounted on the last link of the robotic manipulator such as that detailed in References 1, 2, and 42.

In this case, all of the movements of the bottom platform are related to the fixed top platform, which is fastened to ground. Hence, it is a determination of a mapping that relates the axis through a point of an angle $\delta\underline{\theta}$ to a line of a force increment $\delta\underline{f}$ in the static force applied to the moving body. The mapping is said to refer to a "moving body".

In order to obtain the stiffness matrix for a stiffness mapping, we need to provide six linearly independent small twists to construct a twist matrix according to equation (3.4). For each given twist, the 6-DOF-IPCM reacts the given twist to produce a wrench. Therefore, these six twists produce six wrenches which are linearly independent and in turn define the wrench space.

There are many ways to create a twist and wrench matrix. However, no matter what kind of scheme is selected to produce twist and wrench matrices, the obtained stiffness matrix will be the same or similar. Generally, the ways to generate a twist-and-wrench matrix pair could be categorized into the following three schemes:

(a) Move six individual actuators of the GE-P60 robot sequentially to generate a small twist and record the change in force and torque (i.e. wrench) applied to the system. The forward analysis for the GE-P60 is required in this operation. This is the most simplest and effective way to generate the desired twist and wrench matrices.

(b) Move six combinations of six actuators of the GE-P60 robot sequentially, a combination of two of the GE-P60 actuators, for example, to produce a twist and to record the change in force and torque (i.e. wrench) applied to the system. The forward analysis of the GE-P60 is required in this operation. This operation has no assurance for generating a set of six linearly independent twist vectors. The selected combination of movements need to be carefully selected and checked.

(c) Pre-select six linearly independent small twists and command the GE-P60 to desired positions. This is a reverse operation of either procedure (a) or (b). This operation is done by performing the reverse analysis of the GE-P60 robot to obtain the commanded movement of the GE-P60 robot.

It is worth to note that the GE-P60 robot is at non-singular configuration before and after generating a pair of twist and wrench matrix for all of schemes. In this experiment, the scheme (a) is chosen to generate a twist and wrench matrix. This procedure has the advantage that it generates a set of linearly independent twist vectors and it is easy to monitor the procedure.

With the scheme (a) selected, the movements of the six individual actuators give six linear independent twist changes composed of the total effect of the twist change for the force/torque sensor relative to the ground. During the procedure, six individual encoders embedded with the six actuators track the movement of the GE-P60 robot. It has been shown (see Figure 3.5) that the force/torque sensor is located right at the end of the last link of the robot manipulator. The bottom platform of the 6-DOF-IPCM is secured to the force/torque sensor, and the top platform is combined with the end-effector which is fixed to the ground. Thus, the twist change is representative of the twist change for the grounded end-effector relative to force/torque sensor. In other words, it is the twist change between the top platform and the base.

In the measurement, the resolutions of a force/torque sensor (wrench measurement) and encoders (twist measurement) were considered. In this case, the force resolution was 0.01lb and the moment was 0.01 in-lb. The twist measurement is the twist change applied to the system, which is obtained by calculating the difference between the initial and finial positions and orientations of the end-effector of the manipulator. The commanded actuators of the GE-P60 and the corresponding joint displacements are summarized as follows:

Joints	Actuators Commanded	Joints Commanded
1st	47 4X encoder edges	0.1°
2nd	80	0.1°
3rd	49	0.05°
4th	97	0.2°
5th	63	0.2°
6th	52	0.2°

The commanded twists and the reacting wrenches are dependent upon the stiffness characteristics of the compliant system. In the investigation, the stiffness of a compliant system can not be too soft, otherwise the commanded twists cannot produce reasonable wrenches for measurement. However, a sharp stiffness for a compliant system resulted in that the compliant system cannot produce reasonable twists to be measured and still remain its linearity. The commanded twists are needed to carefully select to provide reasonable twists so that the system can produce reasonable wrenches for measurement without violating the definition of theory of stiffness matrix. The combination of the commanded joint movements summarized as the above is a typical set of twists used in this experiment.

The configuration of the system is set up as illustrated in Figure 3.7. At each sequential measurement, the end-effector remains grounded, and the compliant device should return to its initial (or unloaded) configuration after the movement. The coordinate

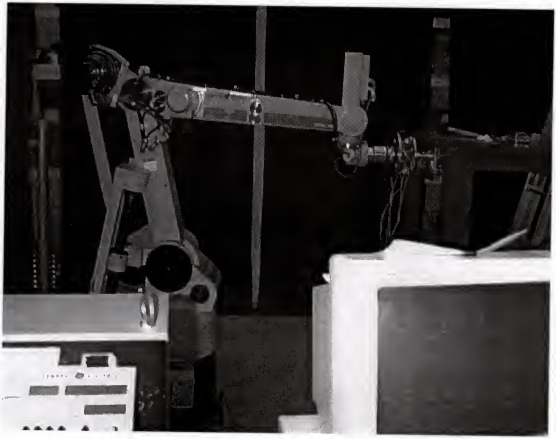

Figure 3.7 The configuration of the GE-P60 was used to generate stiffness matrices. The prototype 6-DOF-IPCM is attached to the GE-P60 as a compliant wrist. The dummy gripper, an aluminum rod, is clamped to the immovable table (the ground).

system chosen to express [K] was located at the center of the force/torque sensor. The z-axis was coaxial with the sixth joint of the robot. The x-axis defined the reference for the last joint of the robot. Assume that the coordinate system is labeled Π. The following is the procedure for the determination of [K] in terms of Π:

(i) The first reading of wrench and twist changes is obtained by keeping the last five actuators fixed while moving the first actuator n encoder values. The coordinates of the wrench and twist change are assumed to be Π. The actuator is then returned to its initial location, and the same command is given again. This process is repeated anywhere from six to ten times. The average of the twist data yields $\delta\underline{D}_1$ and is stored in the first column of the twist matrix $[\delta\underline{D}]$. The average of the wrench data yields $\delta\underline{w}_1$ and is stored in the first column of the wrench matrix $[\delta\underline{w}]$.

(ii) Step (i) was repeated for joints 2 through 6 sequentially, and this yields the twist data $\delta\underline{D}_i$ and wrench data $\delta\underline{w}_i$, (i=2,3,....,6). For each measurement, $\delta\underline{D}_i$ was stored in the i-th column of the matrix $[\delta\underline{D}]$, and $\delta\underline{w}_i$ was stored in the i-th column of the matrix $[\delta\underline{w}]$.

(iii) The 6x6 stiffness matrix [K] was obtained by inverting $[\delta\underline{D}]$ which yields $[K]=[\delta\underline{w}][\delta\underline{D}]^{-1}$ as shown by Equation (3.4).

The procedure to generate a twist matrix and a wrench matrix is illustrated as a flowchart and shown in Figure 3.8. The dependence of a robot configuration on [K] could be examined by changing the configurations of the robotic manipulator and repeating steps (i), (ii) and (iii). The robot configuration was changed by moving some of its joints for certain amounts of encoder values. The combination of these changes generated a new configuration for the robot manipulator. The measure quantified the reliability of the [K]s while the robot was in the distinct configurations.

3.4 Experimental Data

In this investigation, $[\delta\underline{D}]$ and $[\delta\underline{w}]$ are both 6x6 matrices and have full rank; a 6x6 stiffness matrix [K] has thirty-six parameters which can be resolved with an aid of $[\delta\underline{D}]$ and $[\delta\underline{w}]$ by the use of equation (3.4). Let $[\delta\underline{D}]_a$ denote the desired twist matrix and $[\delta\underline{w}]_a$ denote the measured wrench matrix. The GE-P60 robot joints were commanded as shown in the prior section to generate the twists. The obtained twists and resulted wrenches for the each measurement are listed in the Appendix A for reference (Other

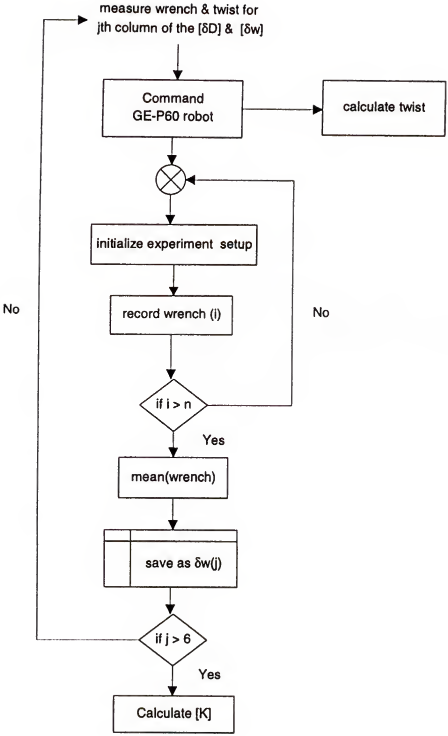

Figure 3.8 Flowchart of the experiment procedure.

combinations were also applied during the investigation and they showed similar results). The one of sets of the obtained twist and wrench matrices is,

$$[\delta \underline{D}]_a = \begin{matrix} -0.00026 & -0.00809 & 0.03745 & -0.02218 & 0.00009 & 0.00000 \\ 0.05834 & -0.00490 & 0.00120 & -0.00193 & -0.02151 & 0.00005 \\ 0.00043 & -0.03906 & 0.00824 & -0.01704 & 0.00003 & 0.00000 \\ -0.00158 & 0.00001 & -0.00001 & -0.00000 & 0.00292 & -0.00001 \\ -0.00001 & 0.00000 & 0.00087 & -0.00299 & 0.00001 & 0.00000 \\ 0.00000 & 0.00000 & -0.00011 & 0.00036 & 0.00000 & -0.00326 \end{matrix}$$

$$[\delta \underline{w}]_a = \begin{matrix} -0.01 & -0.18 & 0.98 & -0.81 & -0.05 & -0.01 \\ 1.58 & -0.20 & 0.10 & -0.15 & -0.78 & 0.00 \\ 0.12 & -2.79 & 0.77 & -1.26 & -0.05 & 0.07 \\ -5.73 & 0.88 & 0.47 & 0.16 & 3.43 & 0.02 \\ 0.07 & -0.51 & 3.09 & -3.18 & -0.09 & -0.01 \\ -0.20 & 0.05 & -0.27 & 0.16 & 0.03 & -0.79 \end{matrix}$$

where each entry in $[\delta \underline{D}]_a$ (or $[\delta \underline{w}]_a$) is the mean value of the raw data shown in category (1), Appendix A. The analysis of this experimental data needs more knowledge of comparison techniques that will be introduced in Chapter 4. Finally, Chapter 5 will present the analysis of the experimental data.

3.5 6-DOC Wrench Control with Theoretical [K]

For the further evaluation, it is required to evaluate the stiffness matrix of the theoretical (or kinematic) model and its capability in controlling the force via displacement. This investigation is carried out by applying six-degrees-of-constraint (6-DOC) wrench control on the experimental instruments illustrated in Figure 3.7. The 6-DOC wrench control is the control strategy to regulate the wrench applied to the end-effector due to a small change in the configuration of the robot manipulator, while the end-effector is fully constrained.

Since a 6-DOF-IPCM is integrated with a robot manipulator as its compliant wrist, it allows the last link of the robot manipulator to have six degrees of freedom for motion.

A small change in the configuration of the robot manipulator results in a wrench increment applied to the end-effector. To accommodate this, the reacting force of the compliant mechanism is induced to keep the system in static equilibrium as the moveable platform twists relative to the ground.

The wrench increment is detected and fed back to the commanding computer. The commanding computer uses this information to generate the corrected twist based upon the control law,

$$\delta \underline{D}_a = -\sigma [K]_t^{-1} \delta \hat{w} \tag{3.5}$$

where σ is a scalar gain (0.04, in this investigation); $\delta\underline{D}_a=[\delta\underline{x}_{oa};\ \delta\underline{\theta}_a]$ is the corrected twist and $\delta\hat{w}$ is the wrench increment at the instant; $[K]_t$ is the theoretically modeled stiffness matrix of the 6-DOF-IPCM and is given by,

$$[K]_t= \begin{array}{cccccc} 23.34 & 0.00 & 0.00 & 10.93 & 84.27 & 0.00 \\ 0.00 & 23.34 & 0.00 & -84.27 & 10.93 & 0.00 \\ 0.00 & 0.00 & 70.80 & 0.00 & 0.00 & -21.86 \\ 10.93 & -84.27 & 0.00 & 489.43 & 0.00 & 0.00 \\ 84.27 & 10.93 & 0.00 & 0.00 & 489.43 & 0.00 \\ 0.00 & 0.00 & -21.86 & 0.00 & 0.00 & 202.26 \end{array}$$

In the matrix, the upper-left 3x3 submatrix has units of "lb/in", the lower-right has units of "lb-in", and the other two 3x3 submatrices have units of "lb". The wrench increment $\delta\hat{w}=[f_x, f_y, f_z\ ;\ m_{ox}, m_{oy}, m_{oz}]$ (or $\delta\hat{w}=[\delta\underline{f}\ ;\ \delta\underline{m}_o]$), is expressed in terms of coordinate system Π, the origin of which is located at the center of a force/torque sensor. The control block diagram is shown in Figure 3.9.

The corrected twist $\delta\underline{D}_a$ is needed to perform a coordinate transformation into $\delta\underline{D}_b=[\delta\underline{x}_{ob};\ \delta\underline{\theta}_b]$ by a 3x3 rotation matrix R_3 before commanding. Hence, the transformation is,

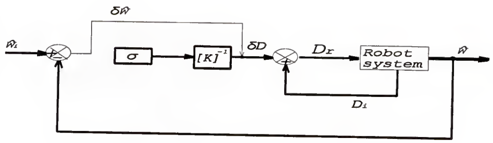

Figure 3.9 The control block diagram of 6-DOC wrench control.

$$\delta\underline{X}_{ob}=R_3\delta\underline{X}_{oa}, \quad \delta\underline{\theta}_b=R_3\delta\underline{\theta}_a. \tag{3.6}$$

The columns of R_3 are the direction cosines of the coordinate axes of Π which are expressed in terms of coordinate system B. The coordinate systems Π and B have the same origins, but the coordinate axes of B are parallel to those grounded systems which is located at the shoulder of the GE-P60 robot.

In each process, the commanding computer performs the reverse and forward kinematic analysis to update the data of the current position and orientation of the end-effector when the new corrected twist is produced. Once the new position and orientation is generated, the commanding computer instructs the robot to the desired configuration.

Figure 3.10 illustrated the change in a wrench during the process. At the beginning, Fy and Mx were the major force and moment applied to the end-effector. The

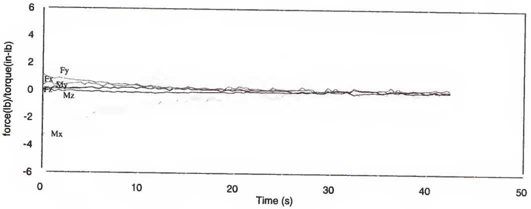

Figure 3.10 A wrench applied to the end-effector is regulated by using the theory of stiffness mapping. The stiffness mapping generates the correcting twist and supplements the robot manipulator for the desired position and orientation according to the control law in Equation (3.5) to nullify the applied wrench. The stiffness matrix was calculated from the mathematical model.

wrench control based upon the control law in equation (3.5) was demonstrated to correct a wrench increment. The process was considered complete when all the applied forces, Fx, Fy, and Fz, and the moments Mx, My, and Mz were zero. The response shown in Figure 3.10 had small disturbances during the process. These disturbances appeared while the control system could not accurately reach the desired position and orientation since there are random errors occurring in the control system, thus the measured wrench was affected. This effect appeared sensitively as the stiffness of the compliant system is high in the direction. As illustrated in Figure 3.10, the strategy was able to manage the errors by feeding these disturbances back to the control system to correct the force/torque (wrench error) by solely controlling the position and orientation of the robot manipulator.

The other two examples were shown in Figures 3.11 and 3.12. These two cases also exhibited similar behaviors as the first example. The control system corrected a wrench increment by using the theory of stiffness mapping.

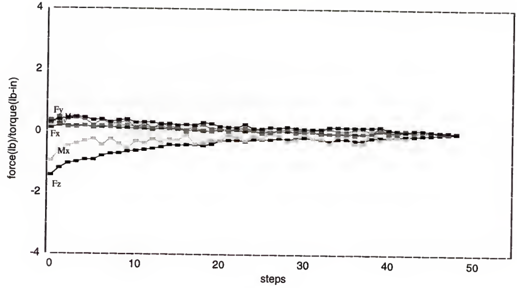

Figure 3.11 A 6-DOC wrench control was demonstrated to correct a wrench applied to the end-effector. It was based upon the control law shown in Equation (3.5). F_z was the major amount of force supply. The stiffness matrix was calculated from the mathematical model.

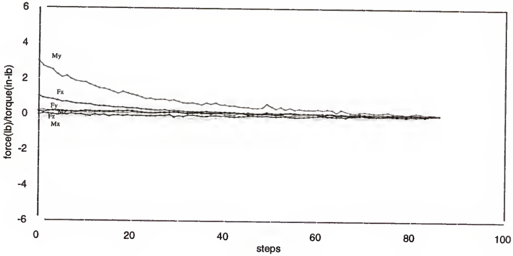

Figure 3.12 A 6-DOC wrench control was demonstrated to correct a wrench applied to the end-effector. It was based upon the control law shown in (3.5). M_y was the major amount of torque applied to the end-effector. The stiffness matrix was obtained from the mathematical model.

This experiment testified the functionality of stiffness mapping in the control the force and position. These cases showed that the stiffness model of the 6-DOF-IPCM is capable of relating twists in a twist space and wrenches in a wrench space. The results also showed that stiffness mapping is an applicable factor on position compensation and force regulation for a robot manipulator to filter the occurring disturbances out during the control process.

3.6 6-DOC Wrench Control with Experimental [K]

The same control strategy presented in Sec.3.5 was also implemented here with an experimental stiffness matrix. The stiffness matrix was taken as the mean of a sample of ten stiffness matrices obtained by experiment. The "mean" measures the average size of the sample so that it could give a better representation of the sample.

This stiffness matrix $[K]_m$ is,

$$[K]_m = \begin{matrix} 24.2930 & -0.8111 & -0.4373 & -22.9802 & 101.3348 & 2.0857 \\ 1.2983 & 25.2215 & 1.5451 & -81.4353 & 12.3581 & 0.1074 \\ 3.6824 & 0.8712 & 71.0986 & -9.2079 & -27.4370 & -21.4375 \\ 19.2653 & -85.6258 & -16.3047 & 546.7408 & -24.9510 & -2.9616 \\ 65.1299 & 0.9939 & -2.3218 & -27.2875 & 552.1290 & -1.0082 \\ -6.0036 & -3.0863 & 0.1179 & -14.3203 & 4.0658 & 232.4443 \end{matrix}$$

This stiffness matrix $[K]_m$ is different to $[K]_t$ as presented in Sec.3.5 when one compares same of the elements of the two matrices. Figure 3.13 illustrates the wrench changes during the process in this case. At the beginning, Fy and Mx were the major force and moment applied to the end-effector. The process was considered complete when the forces, Fx, Fy, and Fz, and the moments Mx, My, and Mz were zero.

The other two examples were as shown in Figure 3.14 and Figure 3.15. These two cases exhibited similar results as shown in the previous example that the control system

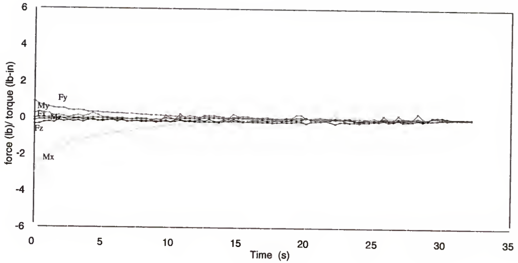

Figure 3.13 A 6-DOC wrench control was demonstrated to correct the wrench applied to the system. Mx was the major amount of torque applied to the end-effector. The stiffness matrix $[K]_m$ was used for this demonstration.

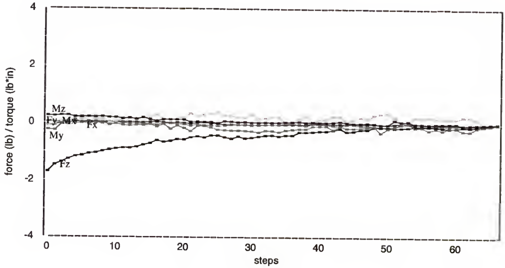

Figure 3.14 A 6-DOC wrench control was demonstrated to correct the wrench applied to the end-effector. It was based upon the control law shown in Equation (3.5). F_z was the major amount of force supply. The stiffness matrix $[K]_m$ was used for this demonstration.

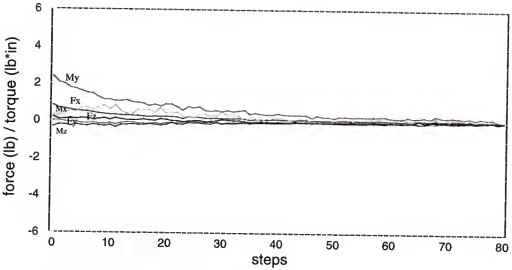

Figure 3.15 A 6-DOC wrench control was demonstrated to correct the wrench applied to the end-effector. It was based upon the control law in (3.5). M_y was the major amount of torque applied to the end-effector. The stiffness matrix $[K]_m$ was used for this demonstration.

regulated a wrench increment by solely controlling the position and orientation of the robot manipulator. These cases showed that the control system was able to manage the wrench errors by feeding these errors back to the control system to gradually correct the force/torque error by changing the position and orientation of the end-effector of the robot manipulator. These results showed that the stiffness matrix is capable of relating a twist in a twist space and a wrench in a wrench space.

CHAPTER 4
METHODOLOGY OF COMPARING GEOMETRIC ENTITIES USING GEOMETRIC INVARIANTS

4.1 Introduction

The goal in this chapter is to introduce the methodology of comparing geometric entities using geometric invariants. This methodology would be the foundation used for verifying the stiffness mapping of a 6-DOF-IPCM in this study. A parameter is said "a geometric invariant" if it is invariant under Euclidean transformation.

As mentioned in Chapter 2, stiffness mapping is a transformation of screws which maps a twist in a twist space to a wrench in a wrench space. Thus, the comparison of two stiffness mappings should be established upon how these respective entities relate to one another by the geometric characteristics of screws. For completely establishing this methodology, we would introduce the comparison of the geometric entities such as points, planes and lines, a comparison of screws then follows. As a preliminary, one shall introduce the comparison of active plane transformations, i.e. two-dimensional mappings which are represented by 3x3 matrices, in Sec.4.4. The scheme of the comparison of active plane transformations is extended from the comparison of two points.

In general, the proximity of matrices can be measured by the use of matrix norms. However, this is only valid when matrix elements have the same dimension. For a stiffness matrix, the matrix elements do not have the same dimension so that a matrix norm cannot be defined. For verifying the stiffness model of a 6-DOF-IPCM, there are two things are considered. One is a mapping's functionality. It is desired to know whether two mappings could map the same twist space to the same wrench space. If they really do, two mappings have the same functionality. The other is that the parameters for comparison should be invariant. Hence, the comparison methodology using geometric invariants is suggested.

It have been showed the ways to obtain the stiffness mapping of a 6-DOF-IPCM theoretically and experimentally. Basically, the verification of the two stiffness models can be achieved by using any of equations defined in (3.2), (3.3) or (3.4). Intuitively, one may desire to compare the theoretical stiffness matrix and the experimental stiffness matrix to verify their agreement. Therefore, the methodology of comparing stiffness matrices using geometric invariant is introduced in Sec.4.5. First, one should identify the invariant properties of a stiffness matrix under Euclidean transformation. These invariant properties of a stiffness matrix have been discussed in Chapter 2, and will be briefly summarized here. Later, one could distinguish the primary properties from those and use them for comparison. The procedure of comparison of two stiffness matrices is thus detailed in Sec.4.5.4. Finally, a discussion for this methodology is presented.

4.2 Points, Planes, Lines

The concept of comparison using geometric invariants is established here by introducing examples of geometric entities (point, plane and line) in three dimensional Euclidean space (E(3)). The geometric invariant relationships between pairs of the entities are given (distance, angle, and distance ⊕ angle, respectively) and subsequently used for comparison.

Consider a point A in E(3) whose homogenous coordinates are $a=[w_A ;x_A ,y_A ,z_A]$[1]. (The coordinates $\mu a=[\mu w_A ;\mu x_A ,\mu y_A ,\mu z_A]$, where $\mu \neq 0$, locate the same point A.) The norm is defined by

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[1] It is assumed here and throughout a Cartesian coordinate system is being used.

$$\|a\| = \sqrt{a^T[\Gamma_p]a}, \tag{4.1}$$

where

$$[\Gamma_p] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and therefore

$$\|a\| = \sqrt{w_a^2} = w_a. \tag{4.3}$$

The coordinates are said to be normalized when $\|a\|=w_a=1$. The position vector $\underline{d}_A$ from origin to A is given by

$$\underline{d}_A = \frac{1}{w_A}(x_A\underline{i}+y_A\underline{j}+z_A\underline{k}). \tag{4.4}$$

The distance of the point A from origin is the magnitude of the position vector $\underline{d}_A$ which is given by

$$\|\underline{d}_A\| = \sqrt{(\frac{x_A}{w_A})^2+(\frac{y_A}{w_A})^2+(\frac{z_A}{w_A})^2} = \sqrt{\underline{d}_A{}^T\,\underline{d}_A}. \tag{4.5}$$

Consider a second point B in E(3) whose homogenous coordinates are $b=[w_B; x_B, y_B, z_B]$ and position vector is $\underline{d}_B$. The distance between the two points is $\|\underline{d}\|$, where

$$\|\underline{d}\| = \|\underline{d}_B-\underline{d}_A\| = \sqrt{(\underline{d}_B-\underline{d}_A)^T(\underline{d}_B-\underline{d}_A)}. \tag{4.6}$$

As $\|\underline{d}\| \to 0$, the two points become the same point.

The distance between two points is invariant under Euclidean transformations. Let $\underline{d}_A'$ be the position vector of point A from the new origin of a new coordinate system, and

$$\underline{d}_A = [R]\underline{d}_A' + \underline{x}, \tag{4.7}$$

where [R] is a 3x3 rotation matrix and $\underline{x}$ is the position vector of the new origin. Then, invariance is demonstrated by substitution of (4.7) into (4.6) for A and B:

$$\begin{aligned} \|\underline{d}\| = \|\underline{d}_B - \underline{d}_A\| &= \|([R]\underline{d}_B' + \underline{x}) - ([R]\underline{d}_A' + \underline{x})\| \\ &= \|[R](\underline{d}_B' - \underline{d}_A')\| = \|\underline{d}_B' - \underline{d}_A'\|. \end{aligned} \tag{4.8}$$

The distance between two points is the only invariant parameter to determine the proximity of two points.

If the two geometrical elements are planes, the only independent parameter considered to compare them is the angle between two planes, β. Consider a plane Π_1 in E(3) whose homogenous coordinates are $\pi_1=[S_1;T_1,U_1,V_1]$. The coordinates are normalized because the following is assumed:

$$\|\pi_1\| = 1, \tag{4.9}$$

where the norm is defined by

$$\|\pi_1\| = \sqrt{S_1^2 + U_1^2 + T_1^2} = \sqrt{\pi_1^T [\gamma_p] \pi_1}, \tag{4.10}$$

where

$$[\gamma_p] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

so that $T_1\underline{i}+U_1\underline{j}+V_1\underline{k}$ is the unit normal vector of the plane, and S_1 is the signed perpendicular distance of the plane from the origin.

Consider a second plane Π_2 whose normalized homogenous coordinates are $\pi_2=[S_2;T_2,U_2,V_2]$. The cosine of the angle β between two planes Π_1 and Π_2 is given by

$$\cos\beta=\pi_1{}^T[\gamma_p]\pi_2. \tag{4.12}$$

As $\cos\beta \rightarrow 1$, the two planes become parallel, and they become the same plane when $S_1=S_2$. As $\cos\beta \rightarrow -1$, the two planes become anti-parallel, and they become the same plane when $S_1=-S_2$. The invariance of (4.12) is analogous to in the that of (4.6).

Next, consider a line G_1 whose homogenous ray coordinates are $g_1=[\underline{s}_1;\underline{s}_{01}]$, which are normalized, i.e. $g_1{}^T[\Gamma]g_1=1$, where

$$[\Gamma]=\begin{pmatrix} I_3 & 0_3 \\ 0_3 & 0_3 \end{pmatrix}. \tag{4.13}$$

The three coordinates $\underline{s}_1$ are the direction cosines of the line, and the three coordinates $\underline{s}_{01}$ are the mutual moments of the line about the x, y, z axes (or the vector cross product $\underline{r}_1 x \underline{s}_1$, where $\underline{r}_1$ is any position vector from origin to the line.) Consider a second line G_2 whose normalized homogenous ray coordinates are $g_2=[\underline{s}_2;\underline{s}_{02}]$. The angle between two lines is α_{12}, and $\cos(\alpha_{12})$ is given by

$$\cos\alpha_{12}=g_1{}^T[\Gamma]g_2. \tag{4.14}$$

The mutual moment of two lines is given by

$$g_1{}^T[\Delta]g_2= -a_{12}\ \sin\alpha_{12}, \tag{4.15}$$

where

$$[\Delta]=\begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}, \tag{4.16}$$

which has been defined in Equation (2.12) and can also be found in Ref.41, and where a_{12} is the signed perpendicular distance between the two lines. The mutual moment is zero if two lines intersect at a finite point, $a_{12} = 0$, or if the two lines are parallel (anti-parallel). Rewriting (4.15) yields an expression for the perpendicular distance,

$$a_{12}=-\frac{g_1{}^T[\Delta]g_2}{|\underline{s}_1 x \underline{s}_2|}, \tag{4.17}$$

where

$$|\underline{s}_1 x \underline{s}_2|=\sin\alpha_{12}. \tag{4.18}$$

Equations (4.14) and (4.18) uniquely determine α_{12}, and when the lines are skew (or intersect at a finite point for which $a_{12}=0$), equation (4.17) uniquely determines a_{12}.

If two lines are parallel, equation (4.17) cannot be used to determine the perpendicular distance between them. For this case, the perpendicular distance a_{12} between two lines can be conveniently determined by using position vectors that are perpendicular to the lines. Suppose that $\underline{d}_1$ and $\underline{d}_2$ are two perpendicular vectors that extend from the origin to the lines G_1 and G_2. (These vectors can be uniquely determined from the expressions $\underline{d}_1=\underline{s}_1 x \underline{s}_{01}$ and $\underline{d}_2=\underline{s}_2 x \underline{s}_{02}$.) Because $\underline{d}_1$ and $\underline{d}_2$ are both perpendicular to $\underline{s}_1$ and $\underline{s}_2$, then so is their difference. Therefore, $|\underline{d}_2-\underline{d}_1|$ is the perpendicular distance between two lines.

As $\alpha_{12} \rightarrow 0$ and $a_{12} \rightarrow 0$, the two lines coalesce (Figure 4.1). This condition shows that two lines are the same line if they have the same direction cosines, and the

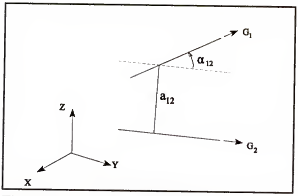

Figure 4.1 Two lines in the space

perpendicular distance between them is zero. Thus, the angle and distance between two lines are two *independent* parameters that should be used to quantify the proximity of the two lines.

Summarizing the above, the comparison of two lines can be carried out by use of equation (4.14) to compute the angle between them, and equation (4.17) to calculate the perpendicular distance between them. The yielded quantities give a measure for the proximity of the two lines.

4.3 Comparison of Two Screws

The concept of comparison using geometric invariants now is extended to screws. The invariant parameters relating two screws are the angle between the screws, the perpendicular distance between the screws and the pitch difference of the screws.

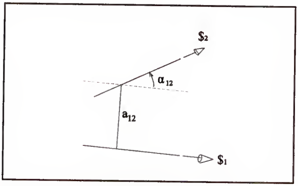

Figure 4.2 Two screws in the space

Consider a screw $\$_1$ whose homogenous ray coordinates are $s_1=[\underline{s}_1;\underline{s}_{01}]$ (Figure 4.2), where the three coordinates $\underline{s}_1$ are the direction cosines and the three coordinates $\underline{s}_{01}$ form the moment vector. The moment vector can be expressed as

$$\underline{s}_{01}=\underline{r}_1 x \underline{s}_1+h_1\underline{s}_1, \qquad (4.19)$$

where $\underline{r}_1$ is any position vector from origin to the line of the screw, and where h_1 is pitch of $\$_1$. The coordinates are normalized because they are chosen to satisfy the following relationship:

$$\|s_1\| = \sqrt{s_1{}^T[\Gamma]s_1} = 1. \tag{4.20}$$

Consider a second screw $\$_2$ whose normalized homogenous ray coordinates are $s_2=[\underline{s}_2;\underline{s}_{02}]$. The reciprocal product of two screws is given by

$$M = s_1{}^T[\Delta]s_2. \tag{4.21}$$

It is now proved that the reciprocal product is invariant under Euclidean transformations. Let s_1' be the coordinates of the screw $\$_1$ in a new coordinate system. Then the pair of coordinates s_1 and s'_1 for $\$_1$ are related by

$$s_1 = [e]s_1', \tag{4.22}$$

where the transformation matrix [e] is given by

$$[e] = \begin{pmatrix} R_3 & 0_3 \\ A_3R_3 & R_3 \end{pmatrix}, \tag{4.23}$$

where R_3 and A_3 are respectively 3x3 rotation and 3x3 skew-symmetric translation matrices. Substituting (4.22) into (4.21) yields

$$M = s_1{}^T[\Delta]s_2 = s_1'^T([e]^T[\Delta][e])s_2'^T. \tag{4.24}$$

Now since

$$[\Delta] = [e]^T[\Delta][e], \tag{4.25}$$

equation (4.24) yields the result

$$M = s_1^T[\Delta]s_2 = s_1'^T[\Delta]s_2'. \tag{4.26}$$

The reciprocal product of two screws may also be expressed in the form,

$$M=(h_1+h_2)\cos\alpha_{12}-a_{12}\sin\alpha_{12}, \tag{4.27}$$

where α_{12} is the angle between $\$_1$ and $\$_2$, and where a_{12} is perpendicular distance between $\$_1$ and $\$_2$. h_1 is the pitch of $\$_1$, and h_2 is the pitch of $\$_2$. The reciprocal product is quantified by three kinds of parameters: h_1 and h_2, α_{12}, and a_{12}. The cosine of angle α_{12} between two screws can be simply obtained by

$$\cos\alpha_{12}=s_1^T[\Gamma]s_2, \tag{4.28}$$

where $[\Gamma]$ is defined by (4.13). The pitches of screws $\$_1$, $\$_2$ can be calculated using

$$h_1=\frac{s_1^T[\Delta]s_1}{2}, \quad h_2=\frac{s_2^T[\Delta]s_2}{2}. \tag{4.29}$$

The perpendicular distance a_{12} between two screws can be obtained by substituting (4.21) into (4.27) and rearranging the form to yield

$$a_{12}=((h_1+h_2)\cos\alpha-(s_1^T[\Delta]s_2))/\sin\alpha. \tag{4.30}$$

The reciprocal product $(s_1^T[\Delta]s_2)$, and angle between two screws α_{12}, and the associated pitches of each screws, h_1, h_2 are invariant under Euclidean transformations.

As $\alpha_{12} \rightarrow 0$, $a_{12} \rightarrow 0$, and the pitch difference $|h_1-h_2| \rightarrow 0$, the two screws coalesce. It is clear that the three scalars (the angle between two screws, their mutual perpendicular distance, and the pitch difference between them) are three *independent* parameters that will be used to quantify how two given screws relate to each other. Consequently, the comparison of two screws could be carried out by the use of the equation (4.28) to obtain the angle between the screws, α_{12}, equation (4.29) to yield the pitch for each screw and then calculate the difference of the pitches $|h_1-h_2|$. The perpendicular distance between two screws, a_{12}, is computed by the use of equation (4.30). The three quantities give us a measure for the proximity of the two unit screws.

4.4 Introduction to The Comparison of Active Plane Transformations

4.4.1 Introduction

In order to clarify the comparison of geometric mappings, we shall start from a plane case. This section introduces the concept of the comparison of geometric mappings in a Euclidean plane by using examples of "active plane transformations (APTs)". The term "active plane transformations" is defined as a transformation that maps a general point in the plane onto a different point in the same plane. The transformation is represented by a 3x3 matrix whose elements are all real.

Let's assume that a point A in the xy plane whose homogenous coordinates are $\underline{a}=[w_a; x_a, y_a]$ is mapped onto another point B in the xy plane whose coordinates are $\underline{b}=[w_b; x_b, y_b]$ by a APT which is given by

$$\underline{b}=[p]\underline{a}, \tag{4.31}$$

where [p] is a non-singular 3x3 transformation matrix of real numbers. The eigenvalue problem of equation (4.31) is $\lambda_i e_i=[p]e_i$, where i=1,2,3. An eigenpoint is defined as a point that is mapped onto itself. The three eigenvectors, e_i, are the coordinates of three eigenpoints of the transformation.

It may be interesting to note the following:

(i) When the eigenvalues are distinct, there are exactly three eigenpoints E_i, which form an invariant triangle.

(ii) When (i) is true, any point other than the three eigenpoints is mapped onto a different point.

(iii) Two of the eigenpoints determine a line which is also invariant. When (i) is true, points on the line which are not eigenpoints are mapped onto different points on the same line.

(iv) When a double root exists, $\lambda_1 = \lambda_2$, then, in general, there is a whole line of eigenpoints, and any two distinct points on the lines may be chosen as E_1 and E_2.

(v) When a triple root occurs, the mapping is, in general, the identity mapping.

(vi) A given mapping has eight independent parameters (three eigenpoints, and ratios of the three eigenvalues.)

(vii) Even though the mapping is real, it may well have complex (or imaginary) eigenvalues and corresponding eigenpoints. When this happens, a reduction in the number of real points mapped onto themselves occurs.

(viii) Degenerate cases may occur where an invariant triangle (real, imaginary, or complex) does not exist [Semple and Kneebone]. Such cases only occur when there are multiple roots. These cases are not considered here.

4.4.2 Comparison and Numerical Examples

There are two examples to illustrate the concept and characteristics of comparing geometric mappings in a Euclidean plane. In these examples, the given matrices are typically sampled and each of the matrices is a variation of one another. The comparison of two given geometric mappings in the xy plane is based upon the invariant eigenpoints and eigenvalues of the mappings. The eigenvalues should be inspected and paired; this in turn establishes how the eigenpoints are paired.

For comparison, the distance between an eigenpoint of one mapping and the corresponding eigenpoint of the other mapping yields one independent parameter. Three distances for three pairs of eigenpoints together with three differences in the eigenvalues establishes the six independent parameters for comparing two geometric mappings of the plane.

Numerical Example 1:

Consider three transformations, APT_1, APT_2, and APT_3 whose matrices are given by the following:

$$[p_1]=\begin{pmatrix} 1 & 1 & 0 \\ -3 & 4 & 1 \\ -3 & 1 & 4 \end{pmatrix},\ [p_2]=\begin{pmatrix} 1.0280 & 0.9801 & 0.0117 \\ -2.9740 & 3.9704 & 1.0331 \\ -2.9919 & 0.9901 & 4.0216 \end{pmatrix},\ [p_3]=\begin{pmatrix} 0.9994 & 0.6840 & 0.3806 \\ -3.2083 & 3.6200 & 1.6083 \\ -3.1006 & 0.6840 & 4.4806 \end{pmatrix}.$$

As an example, consider that the transformations map a general point A whose normalized homogenous coordinates are $\underline{a}$=[1; 8, 2] onto three points B_1, B_2 and B_3, respectively, whose normalized homogenous coordinates are $\underline{b}_1$= [1; 3.4444, 1.4444], $\underline{b}_2$=[1; 3.4699, 1.4588], and $\underline{b}_3$=[1; 4.0052, 1.5669].

The eigenvalues and eigenvectors for $[p_1]$ are

$$[\Lambda_1]=diag[2,3,4]\ ,\ [\Omega_1]=\begin{pmatrix} 2 & -1 & 3 \\ 2 & -2 & 9 \\ 2 & -1 & 9 \end{pmatrix}\ ,$$

which yields the normalized homogenous coordinates for the eigenpoints, [1; 1, 1], [1; 2,1], and [1; 3, 3]. Figure 4.3 shows the locations of the eigenpoints E_1, E_2, and E_3 ,and points A and B_1. The eigenvalues and eigenvectors for $[p_2]$ are

$$[\Lambda_2]=diag[2.99,2.01,4.02]\ ,\ [\Omega_2]=\begin{pmatrix} -1.01 & 2.01 & 2.98 \\ -2.01 & 1.99 & 8.99 \\ -1.0 & 2.01 & 9.01 \end{pmatrix}\ ,$$

which yields the normalized homogenous coordinates for the eigenpoints, [1; 1.9901, 0.9901], [1; 0.9901, 1], and [1; 3.0168, 3.0235]. Finally, the eigenvalues and eigenvectors for $[p_3]$ are

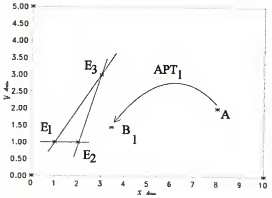

Figure 4.3 Active plane transformation.

$$[\Lambda_3]=diag[4.1,2.9,2.1]\ ,\ [\Omega_3]=\begin{pmatrix} 3.1 & -0.9 & 1.9 \\ 9.1 & -2.0 & 2.0 \\ 8.9 & -0.9 & 1.9 \end{pmatrix}\ ,$$

which yields the normalized homogenous coordinates for the eigenpoints, [1; 2.9355, 2.8709], [1; 2.2222, 1], and [1; 1.0526, 1].

A comparison of the first two mappings APT_1 and APT_2 via pairing the eigen-properties of $[p_1]$ and $[p_2]$ yields the following six quantities:

Quantities	Values
Distance between $E_{p1,1}$ and $E_{p2,2}$	0.01400
Distance between $E_{p1,2}$ and $E_{p2,1}$	0.00995
Distance between $E_{p1,3}$ and $E_{p2,3}$	0.02887
Difference between $\lambda_{p1,1}$ and $\lambda_{p2,2}$	0.01
Difference between $\lambda_{p1,2}$ and $\lambda_{p2,1}$	0.01
Difference between $\lambda_{p1,3}$ and $\lambda_{p2,3}$	0.02

A second comparison of the first and third mappings, APT_1 and APT_3 via pairing the eigen-properties of $[p_1]$ and $[p_3]$ yields the following six quantities:

Quantities	Values
Distance between $E_{p1,1}$ and $E_{p3,3}$	0.14426

Distance between $E_{p1,2}$ and $E_{p3,2}$	0.22222
Distance between $E_{p1,3}$ and $E_{p3,1}$	0.05263
Difference between $\lambda_{p1,1}$ and $\lambda_{p3,3}$	0.1
Difference between $\lambda_{p1,2}$ and $\lambda_{p3,2}$	0.1
Difference between $\lambda_{p1,3}$ and $\lambda_{p3,3}$	0.1

The sum total of eigenvalue differences between the first and second transformation is smaller than the sum total of eigenvalue differences between the first and third. And, the sum total of eigenpoint distances between the first and second transformations is smaller than the sum total of eigenpoint distances between the first and third. Therefore, the second transformation is closer to the first than the third one is.

Numerical Example 2:

Consider three transformations, APT_4, APT_5, and APT_6 whose matrices are given by the following:

$$[p_4]=\begin{pmatrix} 1 & 1 & 0 \\ -3 & 4 & 1 \\ -3 & 1 & 4 \end{pmatrix},[p_5]=\begin{pmatrix} 1.1209 & 0.8018 & 0.1853 \\ -2.9632 & 3.7056 & 1.3736 \\ -2.9790 & 0.8135 & 4.2735 \end{pmatrix},[p_6]=\begin{pmatrix} 0.8969 & 0.8379 & 0.2311 \\ -3.2398 & 38720 & 1.2798 \\ -3.1231 & 0.8379 & 4.2511 \end{pmatrix}.$$

As an example, consider that the transformations map a general point A whose normalized homogenous coordinates are $\underline{a}$=[1; 8, 2] onto three points B_4, B_5 and B_6, respectively, whose normalized homogenous coordinates are $\underline{b}_4$= [1; 3.4444, 1.4444], $\underline{b}_5$=[1; 3.7224, 1.5275], and $\underline{b}_6$=[1; 3.7577, 1.4986].

The eigenvalues and eigenvectors for $[p_4]$ are

$$[\Lambda_4]=diag[2,3,4]\ ,\ [\Omega_4]=\begin{pmatrix} 2 & -1 & 3 \\ 2 & -2 & 9 \\ 2 & -1 & 9 \end{pmatrix}\ ,$$

which yields the normalized homogenous coordinates for the eigenpoints, [1; 1, 1], [1; 2, 1], and [1; 3, 3]. The eigenvalues and eigenvectors for $[p_5]$ are

$$[\Lambda_5]=diag[2.1,2.9,4.1],\ [\Omega_5]=\begin{pmatrix} 2.01 & -1.01 & 2.98 \\ 1.99 & -2.01 & 8.99 \\ 2.01 & -1.0 & 9.01 \end{pmatrix},$$

which yields the normalized homogenous coordinates for the eigenpoints, [1; 0.9901, 1], [1; 1.9901, 0.9901], and [1; 3.0168, 3.0235]. Finally, the eigenvalues and eigenvectors for $[p_6]$ are

$$[\Lambda_6]=diag[2.01,2.99,4.02],[\Omega_6]=\begin{pmatrix} 1.9 & -0.9 & 3.1 \\ 2 & -2 & 9.1 \\ 1.9 & -0.9 & 8.9 \end{pmatrix},$$

which yields the normalized homogenous coordinates for the eigenpoints, [1; 1.0526, 1], [1; 2.2222, 1], and [1; 2.9355, 2.8709].

A comparison of the first two mappings APT_4 and APT_5 via the eigen-properties of $[p_4]$ and $[p_5]$ yields the following six quantities:

Quantities	Values
Distance between $E_{p4,1}$ and $E_{p5,1}$	0.00995
Distance between $E_{p4,2}$ and $E_{p5,2}$	0.01400
Distance between $E_{p4,3}$ and $E_{p5,3}$	0.028867
Difference between $\lambda_{p4,1}$ and $\lambda_{p5,1}$	0.1
Difference between $\lambda_{p4,2}$ and $\lambda_{p5,2}$	0.1
Difference between $\lambda_{p4,3}$ and $\lambda_{p5,3}$	0.1

A second comparison of the first and third mappings, APT_4 and APT_6 via the eigen-properties of $[p_4]$ and $[p_6]$ yields the following six quantities:

Quantities	Values
Distance between $E_{p4,1}$ and $E_{p6,1}$	0.05263

Distance between $E_{p4,2}$ and $E_{p6,2}$	0.22222
Distance between $E_{p4,3}$ and $E_{p6,3}$	0.14426
Difference between $\lambda_{p4,1}$ and $\lambda_{p6,1}$	0.01
Difference between $\lambda_{p4,2}$ and $\lambda_{p6,2}$	0.01
Difference between $\lambda_{p4,3}$ and $\lambda_{p6,3}$	0.02

The answer of which transformation (APT_5 or APT_6) is closer to the first transformation (APT_4) is, for this example, indeterminate. The sum total of eigenvalue differences between the fourth and sixth transformations is smaller than the sum total of eigenvalue differences between the fourth and fifth. However, the sum total of eigenpoint distances between the fourth and fifth transformations is smaller than the sum total of eigenpoint distances between the fourth and sixth. Such a situation is indeterminate because there is no way to assign one value that considers effects of both eigenpoint distances and eigenvalue differences (distance has dimension of length, while the eigenvalue is dimensionless), and one such quantity would be required in this example to make a yes-or-no determination.

Clearly, when more than a single geometrical invariant exists then a comparison may become indeterminate as illustrated by the previous example. A further example of indeterminacy occurs in the comparing three lines G_1, G_2, and G_3 and one can only conclude that G_2 is closer to G_1 when both inequalities $|\alpha_{12}|<|\alpha_{31}|$ and $|a_{12}|<|a_{31}|$ are satisfied. Here, α_{ij} and a_{ij} are the angles between lines G_i and G_j.

4.5 Comparison of Stiffness Matrices

In this section, we present the methodology of comparing two stiffness matrices as a step toward our goal to verify the agreement between a theoretical and a experimental stiffness mappings since the comparison of two stiffness matrices is the most intuitive thought when one examines this issue.

As was the case for points, lines, planes, and screws, a meaningful comparison of two stiffness mappings is made possible by examining those properties (entities) of the individual mappings that are invariant under Euclidean transformations. Since a stiffness mapping is a transformation of screws which maps a twist in a twist space to a wrench in a wrench space, the prime entities for comparison are screws. The comparison model implemented here is based upon quantifying how these respective entities relate to one another.

The columns of a stiffness matrix don't represent screw vectors, thus we can not compare the columns of a given stiffness matrix to the other. The properties of a given stiffness matrix can be found by formulating four distinct eigenvalue problems (see also Chapter 2). It is assumed here that the stiffness matrix (correlation form) is in the most general case, asymmetric with an upper-left 3x3 symmetric submatrix and in the most special case, fully symmetric. Stiffness matrices have been found (experimentally and theoretically) that are asymmetric with symmetric upper-left 3x3 submatrix [1]. Other research has demonstrated that a family of stiffness matrices exists for a given compliant mechanism [40], some of which are symmetric. Therefore, the number of independent parameters for one stiffness matrix has been bracketed to be between twenty-one and thirty-three. The invariant properties of a given stiffness mapping will be extracted from two related forms: the correlation and collineation forms (Secs. 4.5.1, 4.5.2, and 4.5.3). These properties will then be used to relate a pair of given matrices (Sec.4.5.4).

4.5.1 Correlation Form

The stiffness mapping of a 6-DOF in-parallel compliant mechanism have been defined in equation (2.1) which is reproduced in the following,

$$\delta \underline{w} = [K]\delta \underline{D}, \tag{4.40}$$

where $\delta\underline{D}=[\delta\underline{x};\delta\underline{\theta}]$ is the axial coordinates of a small infinitesimal twist and where $\delta\underline{w}=[\delta\underline{f};\delta\underline{m}_0]$ is the ray coordinates of an increment in wrench acting on the system.

It is recognized that signature of [K] prescribes the changing state of potential energy [24]. The signature of [K] is $[\pi,\upsilon,\varsigma]$, where π is the number of positive eigenvalues, and υ is the number of negative eigenvalues, and ς is the number of zero eigenvalues. These eigenvalues are the six roots of the η-equation defined by the determinant[2] $|[K_s]-\eta[I]|=0$, where [I] is the identity matrix. The signature of [K] remains unchanged when any transformation of space occurs (Euclidean or non-Euclidean). Thus, the correlation form gives signature to consider for a comparison of a pair stiffness mappings.

4.5.2 Collineation Form

The stiffness mapping may alternatively be written in the collineation form (as defined in equation (2.14)):

$$\delta \underline{w} = [k]\delta \underline{d}, \tag{4.41}$$

where [k] and [K] are related with the form $[k]=[K][\Delta]$, and $\delta\underline{d}=[\Delta]\delta\underline{D}$. This form is a mapping that transforms a twist having ray coordinates δd into the wrench increment having ray coordinates $\delta\underline{w}$. Comparing (4.41) with (4.40), equation (4.41) combines two successive correlation transformations of screws ([Δ] operator is also a correlation transformation) where the first is ray to axis and the second, axis to ray. Then, the overall transformation is collineation, i.e., ray to ray.

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[2] Only the symmetric part is examined for signature: $[K]_s=([K]+[K]^T)/2$ in this case, since Kinestatic Control requires the symmetric part of a stiffness matrix is positive-definite [1]. This requirement is established from the quadratic forms, $\delta\underline{D}^T[K]\delta\underline{D} = \delta\underline{D}^T[K_s]\delta\underline{D}$.

The six eigenvalues and six eigenscrews of [k] are invariant with any Euclidean transformation. The eigenvalues are given by $|[k]-\lambda[I]|=0$. The ray coordinates of the six eigenscrews satisfy $\lambda_i\hat{s}_i=[k]\hat{s}_i$.

The invariance of eigenvalues and eigenscrews of a given stiffness mapping demonstrated in Chapter 2 shows that [k′] and [k] are similar matrices, which have the same eigenvalues. Application of $\delta d=[e]\delta d'$ to the eigenvectors $\hat{s}_i$ of [k] shows that the eigenvectors $\hat{s}_i'$ of [k′] locate the same six screws, i.e. $\hat{s}_i=[e]\hat{s}_i'$.

Five parameters define a screw in space. The collineation-form eigenvalue problem for a given stiffness mapping yields six eigenvalues together with six eigenscrews, and this results in the thirty-six parameters for the mapping (thirty-three of which must be independent). It is important to note that a stiffness matrix has three positive and three negative eigenstiffnesses (see Sec. 2.6).

4.5.3 Twist- and Wrench-Compliant Axes

The twist- and wrench-compliant axes provide additional means for describing a stiffness mapping. The twist-compliant axes are those twists (3-system) related via the mapping to the pure couples. The wrench-compliant axes are those wrenches (3-system) related via the mapping to the pure translations.

The eigenvalue problem of twist-compliant axes generates three non-zero eigenstiffnesses, μ_1, μ_2, and μ_3, and three corresponding twist-eigenscrews, T_1, T_2, and T_3, which provide a basis for a general three-system. There exists a point p_t which is the center of twist-compliant axes (The origin of the principal screws of the system) [17,18]. A point has three parameters associated, and together with three twist-eigenscrews and three eigenstiffnesses add an additional twenty-one parameters (dependent) to describe the mapping.

Similarly, the eigenvalue problem of wrench-compliant axes generates three non-zero eigenvalues υ_1, υ_2, and υ_3, and three corresponding wrench-eigenscrews, w_1, w_2, and w_3, that provide a basis for a general three-system of screws. There exists a point p_w that is the center of wrench-complaint axes (The origin of the principal screws of the system). (p_t=p_w when [K] is symmetric.) [20,21]. The wrench-eigenscrews together with the center of wrench-complaint axes yield an additional twenty-one parameters.

4.5.4 Comparison

The above invariant properties of a stiffness mapping are summarized in the first column of Table 4.1 and may be used to compare two given stiffness mappings. The right-hand column of Table 4.1 is a list of the number of invariant for the particular category. On the other hand, if we recall Sec.2.6, stiffness matrices usually present certain of special properties and patterns. For example, the symmetric stiffness matrix, its correlation form has six positive signatures, and its collineation form has three positive and three negative eigenvalues. These properties should be recognized as we intend to implement a methodology to compare two stiffness matrices.

The method of comparison for the same type of geometric entities, such as points, planes, lines, and screws, has been introduced in Secs. 4.2 and 4.3. A comparison of two given geometric mappings of the plane has also been presented in Sec.4.4, which relies upon the distance between eigenpoints and eigenvalue differences. Now it remains to use this foundation and apply it to two given stiffness mappings.

Comparison of stiffness mappings may be done in a manner similar to that of active plane transformations (Section 4.4), and this involves using the collineation form

Table 4.1 Summary of the number of invariant properties of a stiffness mapping

Properties	Invariant	# of invariants
1.Correlation	signature of eigenvalues	6
2.Collineation	6 eigenvalues	6
	6 eigenscrews	30
3.TCA	3 eigenvalues	3
	3 eigenscrews	15
Center of TCA	1 point	3
4.WCA	3 eigenvalues	3
	3 eigenscrews	15
Center of WCA	1 point	3

of stiffness mappings to establish a baseline for comparison. It can be said categorically that two given stiffness mappings are equal if the six pairs of eigenvalues[3] of the collineation form are equal and the associated six pairs of eigenscrews are identical (have zero pitch differences, included angles, and perpendicular distances). Therefore a pair of stiffness mappings possesses a maximum of twenty-four independent parameters that may be used for comparison (eighteen for six pairs of eigenscrews, (i.e., six eigenpitches difference, six perpendicular distances and six angles), and six for six pairs of eigenstiffnesses, (i.e., six eigenvalue differences))[4]. The additional meaningful parameters identified in the right-hand column of Table 4.1 should be viewed as redundant but nevertheless, perhaps, useful.

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[3] It is assumed that the eigenvalues of a given stiffness mapping are all real, and there are no such induced eigenscrews that are complex or imaginary.

[4] It is recognized here that the actual number of independent comparison quantities may be lower when restrictions of [K] apply, e.g. symmetry.

At the outset, a comparison of two given stiffness mappings must match the corresponding collineation eigenscrews. The way to pair their eigenscrews is to pair their eigenvalues. Consider λ_1, λ_2 ...., and λ_6 which are the eigenvalues of a mapping, $[k]_\lambda$, and β_1, β_1,...., and β_6 which are the eigenvalues of another one, $[k]_\beta$, respectively. A way to pair these eigrnvalues is to pair their eigenvalues together so that the sum of the absolute values of the magnitude differences $\Sigma|\lambda_i-\beta_j|$, where i and j are not repeated twice, is the smallest.

A given stiffness mapping may have a double root for its characteristic polynomial (see sec. 2.6.2). In this case, the corresponding two eigenscrews forming a cylindroid now collapse into a planar pencil of screws. Any screw in the planar pencil of screws could be chosen as an eigenscrew corresponding to the eigenvalue of the mapping, since the whole planar pencil of screws is preserved. Hence, there exists an eigenscrew in the planar pencil of screws that has the included angle with the paired eigenscrew of the second given stiffness mapping being minimum. The eigenscrew is the correct one to pair with. This eigenscrew must be identified from the planar field of screws so that the comparison is meaningful and the error is eliminated. The eigenscrews can be generated similarly to the one suggested in Reference 42 and introduced as follows.

Suppose that $\$_1$, and $\$_2$ are the two linearly independent eigenscrews corresponding to the eigenvalue, λ_1, of the mapping $[k]_\lambda$. The two eigenscrews form a planar pencil of eigenscrews for the stiffness mapping, $[k]_\lambda$. Any screw $\$_a$ in the planar pencil of screws can be represented as a linear combination of the two screws,

$$\$_a=\zeta_1\$_1+\zeta_2\$_2, \tag{4.42}$$

where ζ_1 and $\zeta_2 \in R$ are scalar multipliers. A unit screw, $\$_p$, being orthogonal to the two screws $\$_1$ and $\$_2$ can be obtained by using Sugimoto-Duffy orthogonalization screws.

Then screws $\$_1$, $\$_2$, and $\$_p$ are three linearly independent screws and form a three-system in E(3).

If ŷ is an eigenscrew of the second stiffness mapping $[k]_\beta$, and corresponds to a certain eigenscrew in the planar pencil of screws, which belongs to $[k]_\lambda$, then, ŷ can be written as

$$\hat{y}=\$_a+\zeta_3\$_p, \tag{4.43}$$

where $\zeta_3 \in R$. Substituting $\$_a$ in (4.42) into (4.43) and rewriting it in matrix form yields the formula,

$$\begin{pmatrix} s_y \\ s_{0y} \end{pmatrix}=\begin{pmatrix} s_1 & s_2 & s_p \\ s_{01} & s_{02} & s_{0p} \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}=\begin{pmatrix} A \\ A_{01} \end{pmatrix}\zeta=\begin{pmatrix} A\zeta \\ A_{01}\zeta \end{pmatrix}, \tag{4.44}$$

where A and A_{01} are 3x3 matrices, and ζ is a 3x1 column matrix, whose elements are scalar multipliers. The scalar multiplier matrix can be determined by taking only the upper three rows of (4.44),i.e. $s_y=A\zeta$. Inverting A yields

$$\zeta = A^{-1}\hat{y}. \tag{4.45}$$

Hence, the picked eigenscrew of $[k]_\lambda$ corresponding to the eigenscrew, ŷ, of $[k]_\beta$ can be obtained by the formula, $\$_a = \zeta_1\$_1 + \zeta_2\$_2$. The scheme can be extended to pick an eigenscrew for the triple root case.

It has been established that pitch differences, angles between screws, and perpendicular distances between screws are invariant under Euclidean transformations. Each comparison of a pair of eigenscrews (one from one mapping correctly related to one from another) gives three invariant parameters to examine their relationship. Six pairs of eigenscrews for two given mappings have eighteen total invariant parameters, which together with six eigenvalue differences give the twenty-four parameters for comparing two stiffness mappings.

It has been stated earlier that determining which of two given geometric mappings is closer to a third does not always yield a definite answer. That statement also applies here. It is only possible to categorically make the determination when the following is true: the sum total of six eigenvalue differences, the sum total of six included angles between two eigenscrews, the sum total of six perpendicular distances between two eigenscrews and the sum total of six pitch differences of two screws are all smaller for one pair of stiffness mappings than they are for the other pair.

It should be noted that the correlation form provides invariant properties of signature for a given stiffness mapping. This can confirm whether or not the changes in state of potential energy for two given stiffness mappings agree.

The three twist-compliant axes and three wrench-compliant axes provide another means for comparison. For example, there are two special points in a stiffness mapping. One is the center of twist-compliant axes, and the other is the center of wrench-compliant axes. Distances between these two pairs of points for the respective mappings provide two of the redundant comparison parameters.

4.6 Discussion

Any meaningful comparison of geometric entities or mappings could not be completed by solely matching coordinates or the numbers in matrices. It is always important to consider, match, and quantify differences in the invariant properties for a given pair of geometric entities or mappings. Table 4.2 summarizes the number of the parameters to locate each of the geometric entities and the number of invariants to compare a pair of them.

Table 4.2 Number of invariant parameters for comparing geometric entities

Entities	# of coordinates	method of comparison
Points	3	Distance
Planes	3	Angle
Lines	4	Angle, Distance
Screws	5	Angle, Distance, Pitch Difference
	# of parameters	# of parameters for comparison
[p]s+	8	6
[K]s*	33	24

+[p] is an active transformation of plane.
*[K] is a stiffness mapping.

The stiffness mapping of a 6-DOF-IPCM is a geometric transformation from a twist space to a wrench space. When one considers comparing their stiffness matrices to verify the agreement between these stiffness mappings, two primary characteristics of matrices should be recognized. One is that the matrices' elements do not have unique dimension, and the other is that the eigenstiffnesses and eigenscrews of stiffness matrices are invariant under Euclidean transformations. Thus, a comparison of two stiffness matrices resolves into a comparison of their geometric entities, screws, and their associated scales. This means that for each pair of eigenstiffnesses and their associated eigenscrews there are four independent parameters involved in the comparison, i.e. one for eigenstiffness and three for eigenscrews (pitch difference, angle and perpendicular distance between screws). These characteristics constitute the backbone for comparing two given stiffness matrices using geometric invariants.

CHAPTER 5
DATA ANALYSIS AND COMPARISON METHODOLOGY IMPLEMENTATION

5.1 Analysis

An analysis on the measurement data, twist and wrench matrices which are shown in Chapter 3, is presented in this chapter. Intuitively, the experimental determination of a stiffness matrix can be achieved by the experimental measurement of a twist-and-wrench matrix pair to calculate the stiffness matrix using equation (3.4). One can thereafter compare the experimental stiffness matrix with the theoretical stiffness matrix to verify their agreement. Unfortunately, the determination of a high-fidelity stiffness matrix is required extremely high resolutions on the determinated twist and wrench matrices. This consequence is illustrated and detailed as the following.

The theory of stiffness mapping is valid when the twist and wrench are small, the twist change and wrench increment thus can be related by Equation (2.1) (or (3.1)). The stiffness matrix is a 6x6 matrix and has full rank. There are thirty-six parameters in the stiffness matrix to be determined when [K] is unknown. The determination of the stiffness matrix [K] for a 6-DOF in-parallel compliant mechanism needs a 6x6 twist matrix and a 6x6 wrench matrix. Each column of the twist matrix $[\delta \underline{D}]$ is a small twist change occurred in the system. Each column of the wrench matrix $[\delta \underline{w}]$ is the wrench increment applied to the system due to the small twist change in the system.

Theoretically, the accuracy in the determination of the stiffness matrix is improved as the twist change approaches the infinitesimal. It also requires that the wrench change is small but not zero. Thus, the resolutions in measuring twist and wrench vectors are two vital factors.

To illustrate this problem, a start is made to consider a similar but simpler case which is the determination of the radius R of a circle by using the equation

$$\delta l = R \times \delta\theta, \tag{5.1}$$

where δθ is an infinitesimal angle and δl is the corresponding chord as illustrated in Figure 6.1. The radius R can be calculated by inverting δθ to yield the equation,

$$R = \delta l \times \delta\theta^{-1}. \tag{5.2}$$

The resulting equation defines the magnitude of angle in radian measure. Since equation (5.2) is valid only when δθ is small, the resolution of the instruments in measuring δθ and the effect of measurement errors on the determination of R are considered.

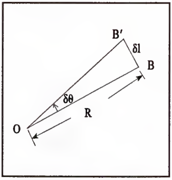

Figure 6.1 Determinating R by use of δθ and δl

Suppose that we desire to measure the radius of a circle. The available apparatus for the measurement in this case are,

(a) the resolution is "0.01 inch" for the length measurement.

(b) the resolution is "0.001 radian" for the plane angle measurement.

There is a circle with a center point O; B and B′ are the two points on the circle. Suppose that we predict the radius of the circle R=OB=10000.00±0.01(inch). The chord length δl =BB′= 1.00±0.01 (inch). Three questions can be posed,

(1)what is the resolution required to measure the small angle δθ between vectors OB and OB′?

(2) if the angle measure between $\underline{OB}$ and $\underline{OB}'$is $\delta\theta_1 = \angle BOB' = 0.01$ (radian), and the radius R_1 is remained 10000.00 inch, what is the choir length δl_1?

(3) if the angle measure between $\underline{OB}$ and $\underline{OB}'$ is $\delta\theta_2 = \angle BOB' = 0.011$(radian), and the chord length for this measurement is $\delta l_2 = 99.99$ (inch), what is the corresponding radius, R_2, determined by $\delta\theta_2$ and δl_2 ? What is the variance between R_1 and R_2 ?

These questions can be answered as the following,

(1) From equation (5.2), $\delta\theta = \angle BOB' = 0.00001$ (radian). The measurement of $\delta\theta$ requires the resolution to be better than tens of micro inch. The magnitude of the $\delta\theta$ meets the condition of the equation (5.1), but it is difficult to measure.

(2)If $\delta\theta_1 = 0.01$(radian), and a predicted radius is $R_1 = 10000.00$ (inch). Thus, the chord length is $BB' = \delta l_1 = 10000.00 * 0.01 = 100.00$(inch).

(3)We know that the resolution in measuring $\delta\theta$ is 0.001. A measure for the small angle is $\delta\theta_2 = 0.011$(radian). A measurement for the chord is $\delta l_2 = 99.99$(inch). Thus, the radius is $R_2 = 99.99 \times (0.011)^{-1} = 9090.00$(inch). The absolute error between R_1 and R_2 is $\varepsilon = |R_1 - R_2| = 910.00$ (inch).

The above example illustrates that the magnitude of the radius R_2 is much different from the true value. The small error in $\delta\theta_2$ results in the error propagation for the determination of R_2. However, we know that the absolute error between δl_1 and δl_2 is only $\xi = |\delta l_1 - \delta l_2| = 0.01$ (inch). The result of the chord length measure is admittable.

This phenomenon is due to the consequence that one tries to use small magnitudes to determine a relatively large one. Any errors in the measurement are sensitively propagated in the calculation of the magnitude of R. This determination of R using equation (5.2) is an ill-conditioned case which the results determined in the measurement are sufficiently sensitive. From this observation, it does not mean that equation (5.2) is invalid although R_1 and R_2 are impossible to be obtained accurately by measurement. We can verify the validation of the results by examining δl in equation (5.1) instead of R in equation (5.2). This situation is the same as what is occurred to the measurement of stiffness matrices.

A 6x6 stiffness matrix associated with a stiffness mapping relates a small twist with a wrench change for a 6-DOF in-parallel compliant mechanism. The stiffness model is valid when a given twist is small that gives a disadvantage in the measurement because the result relies upon the resolution of twist measurements. The smaller twist is desired, the higher resolution for the measurement is needed. Meanwhile, a wrench should be finite but not zero. Moreover, a 6x6 stiffness matrix consists of thirty-six elements, i.e., there are thirty-six parameters needed to be resolved. This case is much more complicated than one-dimensional case to measure the length of a circle's radius.

It has been mentioned earlier in Chapter 3 that the determination of [K] needs six twist-and-wrench pairs, which six twist vectors are linearly independent in twist space, and six wrench vectors are also linearly independent in wrench space. The involvement of multiple pairs of twists and wrenches in the measurements makes the accuracy in the determination of [K] much difficult to reach. This is because any errors in the measurement of twist vectors and wrench vectors cause an error propagation in the determination of a stiffness matrix using equation (3.4). The phenomenon has been discussed preliminarily in the case for the determination of a circle's radius. With this understanding in mind, we shall further investigate the twist error effects on the determination of stiffness matrices.

The Effects of Twist Error on [K]

A small twist error incurring in the twist matrix could crucially result into an error propagation on the determination of stiffness matrix. Technically, the measurement resolution vitally sets the lower bound on the repeatability that one could obtain. This also is the base line for the incurred error in the experiment. Thus, the resolution of the measurement set to be the base line in this investigation. The investigation of this effect can be achieved with an aid of a stiffness matrix which is obtained from kinematic model, $[K]_t$, and a given twist matrix $[\delta\underline{D}]_a$, which has been presented in Sec.3.4. The procedure for this investigation is as the following,

(a) Use the equation $[\delta\underline{w}]_c = [K]_t\,[\delta\underline{D}]_a$ to generate the wrench matrix $[\delta\underline{w}]_c$. This wrench matrix is defined as a true wrench matrix when a twist matrix is given.

(b) Suppose that $[\delta\underline{D}]_a$ is the correct twist matrix. Assume that $\underline{E}$ is a twist error represented in axial coordinates. The column vector $\underline{E}$ is added to each columns of the twist matrix $[\delta\underline{D}]_a$ to generate a new twist matrix, $[\delta\underline{D}]_b$, thus $\delta\underline{D}_{b,i} = \delta\underline{D}_{a,i} + \underline{E}$, where i=1,2,..6, and where $[\delta\underline{D}]_a = \{\delta\underline{D}_{a,1},\ \delta\underline{D}_{a,2}\ \delta\underline{D}_{a,3}, \delta\underline{D}_{a,4}, \delta\underline{D}_{a,5}, \delta\underline{D}_{a,6}\}$, $[\delta\underline{D}]_b = \{\delta D_{b,1}, \delta\underline{D}_{b,2}, \delta\underline{D}_{b,3}, \delta\underline{D}_{b,4}, \delta\underline{D}_{b,5}, \delta\underline{D}_{b,6}\}$. The wrench matrix $[\delta\underline{w}]_c$ is remained unchanged.

(c)Compute the new stiffness matrix $[K]_r$ by using the equation, $[K]_r=[\delta\underline{w}]_c[\delta\underline{D}]_b^{-1}$.

The given stiffness matrix $[K]_t$ and twist matrix $[\delta\underline{D}]_a$ are,

$$[K]_t = \begin{matrix} 23.3400 & 0.0000 & 0.0000 & 10.9285 & 84.2688 & -0.0006 \\ 0.0000 & 23.3400 & 0.0000 & -84.2688 & 10.9285 & 0.0007 \\ 0.0000 & 0.0000 & 70.7999 & 0.0018 & -0.0020 & -21.8569 \\ 10.9285 & -84.2688 & 0.0018 & 489.4321 & 0.0000 & -0.0032 \\ 84.2688 & 10.9285 & -0.0020 & 0.0000 & 489.4321 & -0.0012 \\ -0.0006 & 0.0007 & -21.8569 & -0.0032 & -0.0012 & 202.2579 \end{matrix}$$

$$[\delta\underline{D}]_a = \begin{matrix} -0.00026 & -0.00809 & 0.03745 & -0.02218 & 0.00009 & 0.00000 \\ 0.05834 & -0.00490 & 0.00120 & -0.00193 & -0.02151 & 0.00005 \\ 0.00043 & -0.03906 & 0.00824 & -0.01704 & 0.00003 & 0.00000 \\ -0.00158 & 0.00001 & -0.00001 & -0.00000 & 0.00292 & -0.00001 \\ -0.00001 & 0.00000 & 0.00087 & -0.00299 & 0.00001 & 0.00000 \\ 0.00000 & 0.00000 & -0.00011 & 0.00036 & 0.00000 & -0.00326 \end{matrix}$$

Then, the wrench matrix $[\delta\underline{w}]_c$ is computed by equation, $[\delta\underline{w}]_c = [K]_t\,[\delta\underline{D}]_a$, so that

$$[\delta\underline{w}]_c = \begin{matrix} -0.0242 & -0.1887 & 0.9473 & -0.7696 & 0.0349 & -0.0001 \\ 1.4947 & -0.1152 & 0.0384 & -0.0777 & -0.7480 & 0.0020 \\ 0.0304 & -2.7654 & 0.5858 & -1.2143 & 0.0021 & 0.0713 \\ -5.6924 & 0.3293 & 0.3033 & -0.0798 & 3.2427 & -0.0091 \\ 0.6108 & -0.7352 & 3.5948 & -3.3535 & -0.2226 & 0.0006 \\ -0.0094 & 0.8537 & -0.2024 & 0.4453 & -0.0007 & -0.6594 \end{matrix}$$

In order to clearly inspect the twist error effect on the determination of stiffness matrix, one should individually examine the position and orientation error effect. Let us rewrite a twist error vector as a combination of a position and orientation vectors, i.e $\underline{E} = [\delta\underline{x} ; \delta\underline{\theta}] = [\delta\underline{x} ; \underline{0}] + [\underline{0} ; \delta\underline{\theta}]$, where $\underline{0}$ is 3x1 zero vector. Thus a twist vector $\underline{E} = [\delta\underline{x} ; \underline{0}] = [\delta x_x,\ \delta x_y, \delta x_z; 0.0, 0.0, 0.0]$[1] represents that a position error $\delta\underline{x}$ occurs in the measurement, for example. As the representation, the position and orientation error effects can be observed individually. One first investigates the position error effect on the determination of stiffness matrices.

It is given the GE-P60's repeatability is ±0.008inch (see GE-P60 Manual [38]) . The quantity is taken as the reference position error in this investigation. Thus, if the twist error vector is specified, the stiffness matrix can be computed directly from procedures (b) and (c).

(i) If $\underline{E} = [\delta\underline{x};\delta\underline{\theta}] = [0.001, 0.001, 0.001; 0.000, 0.000, 0.000]$, the stiffness matrix generated with this variation built into the twist matrix is,

$$[K]_{r1} = \begin{matrix} 22.3247 & -0.7526 & 0.8760 & -2.3199 & 95.6340 & 6.9144 \\ -1.0153 & 22.5874 & 0.8760 & -97.5172 & 22.2937 & 6.9157 \\ -3.0798 & -2.2830 & 73.4572 & -40.1862 & 34.4735 & -0.8808 \\ 14.1188 & -81.9039 & -2.7508 & 531.0611 & -35.7116 & -21.7314 \\ 80.1278 & 7.8589 & 3.5709 & -54.0355 & 535.7867 & 28.2026 \\ 0.9502 & 0.7055 & -22.6772 & 12.4033 & -10.6442 & 195.7823 \end{matrix}$$

(ii) If $\underline{E} = [\delta\underline{x};\delta\underline{\theta}] = [0.0001, 0.0001, 0.0001; 0.000, 0.0000, 0.0000]$, the stiffness matrix generated with this variation built into the twist matrix is

$$[K]_{r2} = \begin{matrix} 23.2349 & -0.0779 & 0.0907 & 9.5565 & 85.4458 & 0.7155 \\ -0.1051 & 23.2621 & 0.0907 & -85.6408 & 12.1055 & 0.7168 \\ -0.3190 & -0.2364 & 71.0751 & -4.1601 & 3.5683 & -19.6846 \\ 11.2589 & -84.0239 & -0.2833 & 493.7433 & -3.6984 & -2.2534 \\ 83.8399 & 10.6106 & 0.3680 & -5.5960 & 494.2327 & 2.9196 \\ 0.0979 & 0.0737 & -21.9419 & 1.2816 & -1.1034 & 201.5873 \end{matrix}$$

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[1] The error vector (either twist or wrench vectors) represented in this ways are considered as a column vector which is equivalent to the expression $[\delta x_x, \delta x_y, \delta x_z; 0.0, 0.0, 0.0]^T$.

(iii) If $\underline{E}$= $[\delta \underline{x};\delta \underline{\theta}]$=[0.00005, 0.00005, 0.00005; 0.0000, 0.00000, 0.00000], the stiffness matrix generated with this variation built into the twist matrix is

$$[K]_{r3} = \begin{matrix} 23.2873 & -0.0390 & 0.0455 & 10.2411 & 84.8585 & 0.3582 \\ -0.0527 & 23.3010 & 0.0455 & -84.9562 & 11.5182 & 0.3595 \\ -0.1598 & -0.1184 & 70.9378 & -2.0833 & 1.7867 & -20.7686 \\ 11.0940 & -84.1461 & -0.1410 & 491.5920 & -1.8528 & -1.1305 \\ 84.0539 & 10.7692 & 0.1834 & -2.8036 & 491.8371 & 1.4621 \\ 0.0487 & 0.0373 & -21.8995 & 0.6405 & -0.5534 & 201.9219 \end{matrix}$$

(vi) If $\underline{E}$= $[\delta x;\delta\theta]$=[0.00001, 0.00001, 0.00001; 0.0000, 0.00000, 0.00000], the stiffness matrix generated with this variation built into the twist matrix is

$$[K]_{r4}= \begin{matrix} 23.3294 & -0.0078 & 0.0091 & 10.7908 & 84.3869 & 0.0713 \\ -0.0106 & 23.3322 & 0.0091 & -84.4065 & 11.0466 & 0.0726 \\ -0.0320 & -0.0237 & 70.8275 & -0.4159 & 0.3563 & -21.6389 \\ 10.9617 & -84.2442 & -0.0268 & 489.8648 & -0.3712 & -0.2290 \\ 84.2258 & 10.8966 & 0.0351 & -0.5616 & 489.9139 & 0.2919 \\ 0.0093 & 0.0080 & -21.8654 & 0.1257 & -0.1118 & 202.1906 \end{matrix}$$

To identify the effects of orientation errors on the determination of stiffness matrices, a similar process as the above can be used. The variation twist is set to be $\underline{E}$= $[\delta x;\delta\theta]$=[0.0000, 0.0000, 0.0000; $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$] which represents the orientation error incurred in a twist. For conducting this investigation, the orientation error of the twist in an axis is estimated. This estimation is based upon the GE-P60 robot's repeatability causing an orientation error in the mechanism. The repeatability 0.008(inch) of the GE-robot can be represented by a sphere with a diameter 0.008inch. The bottom platform of the 6-DOF-IPCM is attached to the end point of robot manipulator (Figure 5.2a), so that we may assume that the

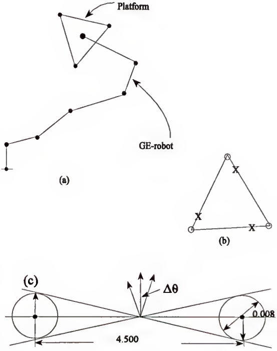

Figure 5.2

mechanism has the same repeatability as the GE-robot. The mechanism's six pivot points are on the same plane and determine the orientation of the platform. However, three points are enough to determine a plane. The selected pivot points marked by "○" are as shown in Figure 5.2b. The distance between a pair of these pivot points is 4.500 inch. Assume that every pivot points on the plane have the same repeatability. The maximum offset is when the pivot points are moved to opposite direction to have an 0.008inch offset (Figure 5.2c), so that the normal vector of the plane is rotated about $\Delta\theta = \sin^{-1}(0.008/4.500) = 0.002$ (radian). For the investigation of the orientation error effects on the determination of stiffness matrices, we use this estimation as the reference orientation error to set the bounded variation in these cases. The regenerated stiffness matrices are shown as the following when the error vector is given.

(i) If $\underline{E}= [\delta x;\delta\theta]=[0.000, 0.000, 0.000; 0.001, 0.001, 0.001]$, the stiffness matrix which is generated with this variation built into the twist matrix is given as

$$[K]_{o1} = \begin{matrix} 17.7904 & -4.1138 & 4.7883 & -61.4869 & 146.3907 & 37.7966 \\ 4.2754 & 26.5093 & -3.6889 & -28.4799 & -36.9303 & -29.1183 \\ 1.2742 & 0.9445 & 69.7005 & 16.6283 & -14.2651 & -30.5351 \\ -17.6032 & -105.4187 & 24.6194 & 117.1272 & 319.3835 & 194.3212 \\ 55.7370 & -10.2215 & 24.6157 & -372.3064 & 808.8169 & 194.3240 \\ -11.7912 & -8.7394 & -11.6838 & -153.8559 & 131.9821 & 282.5613 \end{matrix}$$

(ii) If $\underline{E}= [\delta x;\delta\theta]=[0.0000, 0.0000, 0.0000; 0.0001, 0.0001, 0.0001]$, the stiffness matrix which is generated with this variation built into the twist matrix is given as

$$[K]_{o2} = \begin{matrix} 22.8996 & -0.3265 & 0.3800 & 5.1813 & 89.1990 & 2.9991 \\ 0.3393 & 23.5915 & -0.2928 & -79.8412 & 7.1303 & -2.3103 \\ 0.1011 & 0.0750 & 70.7126 & 1.3213 & -1.1340 & -22.5456 \\ 8.6641 & -85.9473 & 1.9555 & 459.8846 & 25.3474 & 15.4191 \\ 82.0044 & 9.2500 & 1.9518 & -29.5476 & 514.7796 & 15.4212 \\ -0.9363 & -0.6929 & -21.0495 & -12.2135 & 10.4735 & 208.6311 \end{matrix}$$

(iii) If $\underline{E}= [\delta x;\delta\theta]=[0.00000, 0.00000, 0.00000; 0.00001, 0.00001, 0.00001]$, the stiffness matrix which is generated with this variation built into the twist matrix is given as

$$[K]_{o3} = \begin{matrix} 23.2968 & -0.0320 & 0.0372 & 10.3654 & 84.7519 & 0.2933 \\ 0.0332 & 23.3646 & -0.0287 & -83.8350 & 10.5564 & -0.2257 \\ 0.0099 & 0.0073 & 70.7914 & 0.1311 & -0.1129 & -21.9244 \\ 10.7066 & -84.4333 & 0.1932 & 486.5371 & 2.4835 & 1.5078 \\ 84.0469 & 10.7640 & 0.1894 & -2.8950 & 491.9156 & 1.5099 \\ -0.0923 & -0.0673 & -21.7778 & -1.1995 & 1.0251 & 202.8823 \end{matrix}$$

This examination shows that the subjected errors in the twists sensitively distribute an error propagation to a stiffness matrix. To verify the results, the other combinations of twist vectors are investigated for twist error affecting the result of the determination of stiffness matrices. They showed similar results.

The effects of Wrench Error On [K]

A small wrench error occurred in the measured wrench matrix has an essential effect on the determination of a stiffness matrix due to the error propagation in the calculation. Technically, the measurement resolution vitally sets the lower bound on the repeatability that one could obtained. This also is the base line for the incurred error in the experiment. Thus, the resolution of the measurement is the base line in this investigation. We have been given a twist matrix $[\delta\underline{D}]_a$ and stiffness matrix $[K]_t$, and a wrench matrix $[\delta\underline{w}]_c$ is computed from the equation, i.e. $[\delta\underline{w}]_c = [K]_t\,[\delta\underline{D}]_a$ in prior section. We may make use these three matrices to continue this investigation. The investigation can be achieved by adding a wrench error vector $\upsilon=[\delta\underline{f};\ \delta\underline{m}]$ to each columns of the wrench matrix $[\delta\underline{w}]_c$ and recalculating the stiffness matrix . If the new wrench matrix is denoted as $[\delta\underline{w}]_\upsilon$, and $\delta\underline{w}_{\upsilon,i}$ is its i-th column vector, where i=1,2,3,...6, then $\delta\underline{w}_{\upsilon,i} = \delta\underline{w}_{c,i} + \upsilon$, here $\delta\underline{w}_{c,i}$ is denoted as the i-th column vector of the wrench matrix $[\delta\underline{w}]_c$. The new stiffness matrix due to a wrench error built into the wrench matrix is obtained by use of the equation $[K]_\upsilon = [\delta\underline{w}]_c\,[\,\delta\underline{D}]_a^{-1}$. This procedure is similar to the above section but the twist matrix is remained unchanged in this case. We can decompose the wrench error into the force error and moment error. First, the force error effect on the determination of stiffness matrices is conducted. It is known that the resolution of the force measurement is 0.01lb. Let's assumed that the error in the force measurement is bounded within 0.1lb. Then, we increase the resolution for each investigation by one of tenth of the previous one. The results are shown as the following when the wrench error vector is given.

(i) If $\upsilon=[\delta\underline{f}; \delta\underline{m}]=[0.1, 0.1, 0.1; 0.0, 0.0, 0.0]$, the stiffness matrix generated with this wrench error vector built into the wrench matrix is ,

$$[K]_{\upsilon 1} = \begin{matrix} 27.8629 & 3.3527 & -3.9024 & 69.9466 & 33.6399 & -30.8051 \\ 4.5229 & 26.6927 & -3.9024 & -25.2507 & -39.7004 & -30.8038 \\ 4.5229 & 3.3527 & 66.8975 & 59.0199 & -50.6309 & -52.6614 \\ 10.9285 & -84.2688 & 0.0018 & 489.4321 & -0.0000 & -0.0032 \\ 84.2688 & 10.9285 & -0.0020 & 0.0000 & 489.4321 & -0.0012 \\ -0.0006 & 0.0007 & -21.8569 & -0.0032 & -0.0012 & 202.2579 \end{matrix}$$

(ii) If $\upsilon=[\delta\underline{f}; \delta\underline{m}]=[0.01, 0.01, 0.01; 0.00, 0.00, 0.00]$, the stiffness matrix generated with this wrench error vector built into the wrench matrix is ,

$$[K]_{\upsilon 2} = \begin{matrix} 23.7923 & 0.3353 & -0.3902 & 16.8303 & 79.2059 & -3.0810 \\ 0.4523 & 23.6753 & -0.3902 & -78.3670 & 5.8656 & -3.0797 \\ 0.4523 & 0.3353 & 70.4097 & 5.9036 & -5.0649 & -24.9373 \\ 10.9285 & -84.2688 & 0.0018 & 489.4321 & -0.0000 & -0.0032 \\ 84.2688 & 10.9285 & -0.0020 & 0.0000 & 489.4321 & -0.0012 \\ -0.0006 & 0.0007 & -21.8569 & -0.0032 & -0.0012 & 202.2579 \end{matrix}$$

(iii) If $\upsilon=[\delta\underline{f}; \delta\underline{m}]=[0.001, 0.001, 0.001; 0.000, 0.000, 0.000]$, the stiffness matrix generated with this wrench error vector built into the wrench matrix is ,

$$[K]_{\upsilon 3} = \begin{matrix} 23.3852 & 0.0335 & -0.0390 & 11.5187 & 83.7625 & -0.3086 \\ 0.0452 & 23.3735 & -0.0390 & -83.6786 & 10.4222 & -0.3073 \\ 0.0452 & 0.0335 & 70.7609 & 0.5920 & -0.5083 & -22.1649 \\ 10.9285 & -84.2688 & 0.0018 & 489.4321 & 0.0000 & -0.0032 \\ 84.2688 & 10.9285 & -0.0020 & 0.0000 & 489.4321 & -0.0012 \\ -0.0006 & 0.0007 & -21.8569 & -0.0032 & -0.0012 & 202.2579 \end{matrix}$$

We have presented the force error effects on the determination of stiffness matrices . In the following, the moment error effect on the determination of stiffness matrices is illustrated. It is known that the resolution of the moment measurement is 0.01inch-lb. Let's increase the error in the measurement a little more than the resolution to start with the moment error within 0.1inch-lb. The results are shown as the following when the wrench error vector is given.

(i) If $\upsilon=[\delta\underline{f};\ \delta\underline{m}]=[0.0, 0.0, 0.0;\ 0.1, 0.1, 0.1]$, the stiffness matrix generated with this wrench error built into the wrench matrix is ,

$$[K]_{\upsilon 4} = \begin{matrix} 23.3400 & -0.0000 & -0.0000 & 10.9285 & 84.2688 & -0.0006 \\ -0.0000 & 23.3400 & -0.0000 & -84.2688 & 10.9285 & 0.0007 \\ -0.0000 & 0.0000 & 70.7999 & 0.0018 & -0.0020 & -21.8569 \\ 15.4514 & -80.9161 & -3.9006 & 548.4502 & -50.6289 & -30.8077 \\ 88.7917 & 14.2812 & -3.9044 & 59.0181 & 438.8032 & -30.8057 \\ 4.5223 & 3.3534 & -25.7593 & 59.0149 & -50.6301 & 171.4534 \end{matrix}$$

(ii) If $\upsilon=[\delta\underline{f};\ \delta\underline{m}]=[0.00, 0.00, 0.00;\ 0.01, 0.01, 0.01]$, the stiffness matrix generated with this wrench error built into the wrench matrix is ,

$$[K]_{\upsilon 5} = \begin{matrix} 23.3400 & -0.0000 & -0.0000 & 10.9285 & 84.2688 & -0.0006 \\ -0.0000 & 23.3400 & -0.0000 & -84.2688 & 10.9285 & 0.0007 \\ -0.0000 & 0.0000 & 70.7999 & 0.0018 & -0.0020 & -21.8569 \\ 11.3808 & -83.9335 & -0.3884 & 495.3339 & -5.0629 & -3.0836 \\ 84.7211 & 11.2638 & -0.3922 & 5.9018 & 484.3692 & -3.0816 \\ 0.4517 & 0.3360 & -22.2471 & 5.8986 & -5.0641 & 199.1775 \end{matrix}$$

(iii) If $\upsilon=[\delta\underline{f};\ \delta\underline{m}]=[0.000, 0.000, 0.000;\ 0.001, 0.001, 0.001]$, the stiffness matrix generated with this wrench error built into the wrench matrix is ,

$$[K]_{\upsilon 6} = \begin{matrix} 23.3400 & -0.0000 & -0.0000 & 10.9285 & 84.2688 & -0.0006 \\ -0.0000 & 23.3400 & -0.0000 & -84.2688 & 10.9285 & 0.0007 \\ -0.0000 & 0.0000 & 70.7999 & 0.0018 & -0.0020 & -21.8569 \\ 10.9737 & -84.2353 & -0.0372 & 490.0223 & -0.5063 & -0.3112 \\ 84.3140 & 10.9620 & -0.0410 & 0.5902 & 488.9258 & -0.3092 \\ 0.0446 & 0.0342 & -21.8959 & 0.5870 & -0.5075 & 201.9499 \end{matrix}$$

This examination shows the characteristics of an error propagation in computing a stiffness matrix when the measured wrenches are subjected to the measurement errors. To verify this result, the other combinations of wrench vectors are investigated for wrench errors affecting the result of the determination of stiffness matrices. They show similar results.

Remarks

The above cases investigate the effects of the twist and wrench errors on the determination of stiffness matrices. Both the twist and wrench measurement required a much higher resolution than expected.

It is difficult to get the accuracy in the measurement of stiffness matrices since the current facility cannot satisfy the required resolution. These errors in the six twist vector and six wrench vectors result in the error propagation when one compute the desired stiffness matrix. Because of this situation, the obtained stiffness matrix is thus the result with an error which is generated from both of the twist and wrench errors.

However, in our measurement, the stiffness matrix is like a black box which is unknown inside. An input twist produces a wrench change. From this relationship, a stiffness matrix which relates a twist and a wrench is defined. Thus, a verification of the experimental result for the kinematic model can be directed to the verification of the wrench matrices. In the following section, a definition of equivalent stiffness matrices is given. The definition establishes the philosophy to verify the measuring results by the definition of equivalent wrench matrices.

5.2 Equivalent Stiffness Matrices

In this section, we discuss the definition of equivalent stiffness matrices. The definition of equivalent stiffness matrices leads to establish the equivalent wrenches, which will be discussed in next section. Two stiffness matrices are said to be equivalent if two wrench matrices are equivalent when the same twist matrix is applied to a compliant system. These equivalent stiffness matrices characterize completely the effect of the given mapping on a 6-DOF-IPCM. The definition is established upon the physical meaning of the equivalent systems of forces and is presented in the following.

For a 6-DOF-IPCM, a stiffness mapping, $\boldsymbol{S}: \boldsymbol{D} \rightarrow \boldsymbol{W}$, where $\boldsymbol{D}$ is the twist space and $\boldsymbol{W}$ is the wrench space; both are six-dimensional vector spaces. This one-to-one relationship between a small twist and a finite wrench is as defined in equation (2.1) which is

$$\delta \underline{w} = [K]\delta \underline{D}. \tag{5.3}$$

From (5.3), if a system generates a small twist change, then a wrench increment is obtained. The relationship between the twist and wrench is related by stiffness matrix [K].

Let $[\delta \underline{D}]$ be an ordered basis for the twist vector space $\boldsymbol{D}$,

$$[\delta \underline{D}] = \{\delta \underline{D}_1\ \delta \underline{D}_2\ \delta \underline{D}_3\ \delta \underline{D}_4\ \delta \underline{D}_5\ \delta \underline{D}_6\}. \tag{5.4}$$

$[\delta \underline{D}]$ is a set of six small twist vectors which are linearly independent in six-dimensional vector space. The six twist vectors completely span a twist space. Any twist in this twist space can be written as a linear combination of these six twists. At the instance, suppose that one supplies a small twist $\delta \underline{D}_i$, i=1,2,3..,6, where $\delta \underline{D}_i$ is a vector in the twist space $\boldsymbol{D}$, to the system. We have six wrench vectors defined as

$$\delta \underline{w}_i = [K]\delta \underline{D}_i \ \ where\ \ i=1,2,3..,6. \tag{5.5}$$

From equations (5.4) and (5.5), a wrench matrix is yielded. Then the twist matrix and the wrench matrix can be combined together and written as,

$$[\delta \underline{w}] = [K][\delta \underline{D}], \tag{5.6}$$

where $[\delta \underline{D}]$ is a 6x6 twist matrix as defined in (5.4), and $[\delta \underline{w}]$ is a 6x6 wrench matrix defined as,

$$[\delta \underline{w}] = \{\delta \underline{w}_1 \;\; \delta \underline{w}_2 \;\; \delta \underline{w}_3 \;\; \delta \underline{w}_4 \;\; \delta \underline{w}_5 \;\; \delta \underline{w}_6\}. \tag{5.7}$$

Since S is one-to-one mapping, $[\delta \underline{w}]$ is composed of six wrench vectors which are linearly independent in six-dimensional vector space. These six wrenches form a basis for a wrench space. Any wrench in this wrench space can be written as a linear combination of these six wrenches. Thus, the wrench space is completely defined.

When $[\delta \underline{w}]$ and $[\delta \underline{D}]$ are both specified, the stiffness matrix [K] can be determined by inverting $[\delta \underline{D}]$ in Equation (5.6). Thus,

$$[K] = [\delta \underline{w}][\delta \underline{D}]^{-1} \;, \tag{5.8}$$

which is the same equation defined in (3.3).

Suppose that we have two stiffness matrices, $[K]_a$, an experimentally measured one, and $[K]_t$, a theoretical one. A twist matrix, $[\delta \underline{D}]_g$ is given. From equation (5.6), the two wrench matrices can be obtained as,

$$[\delta \underline{w}]_t = [K]_t \, [\delta \underline{D}]_g. \tag{5.9}$$

$$[\delta \underline{w}]_a = [K]_a \, [\delta \underline{D}]_g. \tag{5.10}$$

The two stiffness matrices are said to be equivalent when the wrench matrices, $[\delta \underline{w}]_t$ and $[\delta \underline{w}]_a$, are equivalent.

However, this definition for the equivalent stiffness matrices must be modified slightly in its application because a measurement of twist matrix and wrench matrix usually are considered within certain admittable errors. If a twist matrix, $[\delta \underline{D}]_g$, is given, the two stiffness matrices, $[K]_t$ and $[K]_a$, are said to be equivalent when the two wrench matrices are related by $[\delta \underline{w}]_a = [\delta \underline{w}]_t + [\underline{\varepsilon}]$, where $[\underline{\varepsilon}]$ is a 6x6 error matrix.

In this theorem, the twist matrix is assumed to be given in compliance with the definition of stiffness mapping, which is a small twist subjected to the system. This theorem offers a way to verify two stiffness matrices by comparing two wrench matrices. Since using equation (5.8) to compute the desired stiffness matrix is an ill-conditioned case due to the errors propagated in the calculation, the result is not appropriate for the comparison in this case. However, two wrench matrices, which one is generated from experimental measurement, and the other is obtained from theoretical calculation, can be used to a comparison. The propagated effects in the calculation have been eliminated.

5.3 Equivalent Wrenches

The preceding section shows the definition of equivalent stiffness matrices, which is based upon the physical meaning of equivalent systems of forces. An equivalent wrench is physically equal to a system of forces. They are only different with the representation. For a rigid body, two systems of forces $\underline{f}_1$, $\underline{f}_2$, $\underline{f}_3$, $\underline{f}_4$, $\underline{f}_5$, $\underline{f}_6$, etc., and $\underline{f}'_1$, $\underline{f}'_2$, $\underline{f}'_3$, $\underline{f}'_4$, $\underline{f}'_5$, $\underline{f}'_6$, etc., are equivalent if, and only if, the sums of the forces and sums of moments about a given point O of the forces of the two systems are, respectively equal. Expressed mathematically, the necessary and sufficient condition for the two systems of forces to be equivalent are

$$\Sigma \underline{f} = \Sigma \underline{f}' \ \ and \ \ \Sigma \underline{m}_0 = \Sigma \underline{m}'_0. \tag{5.11}$$

The equation have a simple physical significance. It express that two systems of forces are equivalent if they tend to impart to the rigid body (1)the same translation in the direction of acting forces, and (2) the same rotation about the axis of acting moments.

In the general case of a system of forces in space, the equivalent force-moment system at O consists of a force $\underline{f}$ and a couple vector $\underline{m}^f_o$ which are not perpendicular, and neither of which is zero (Figure 5.3a). Thus, the system of forces cannot be reduced to a single force or a single couple. The couple vector may be decomposed into two couple vectors which are a component $\underline{m}_1$ along $\underline{f}$ and a component $\underline{m}_2$ in a plane perpendicular to $\underline{f}$ (Figure 5.3b). The force $\underline{f}$ and the couple vector $\underline{m}_2$ may be then replaced by a single force $\underline{f}$ acting along a new line of action.

The original system of forces thus reduces to $\underline{f}$ and to the couple vector $\underline{m}_1$ (Figure 5.3c), i.e., to $\underline{f}$ and a couple acting in the plane perpendicular to $\underline{f}$. Thus perpendicular force-couple system is called a wrench. The line of action of $\underline{f}$ is known as the axis of the wrench, and the ration $h=|\underline{m}_1|/|\underline{f}|$, where $|\underline{m}_1|$ is the magnitude of $\underline{m}_1$, and $|\underline{f}|$ is the magnitude of $\underline{f}$, is called the pitch of the wrench. A wrench , therefore, can be written as the combination of two collinear vectors, namely, a force $\underline{f}$ and a couple vector $\underline{m}_1=h\underline{f}$, i.e.

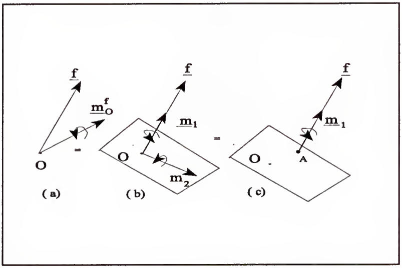

Figure 5.3 A system of forces.

$$\hat{w}=[\underline{f}\ ;\ \underline{m}_0]=[\underline{f}\ ;\ \underline{r}\times\underline{f}]+[\underline{0}\ ;\ h\underline{f}], \tag{5.12}$$

Where, $\underline{r}$ is a position vector form the referred point O to the line $\underline{f}$, and h is the pitch of the wrench.

A system of forces can be represented as a force-couple pair, or a wrench. The force-couple pair is equivalent to a wrench. A comparison of two force-couple pairs is equivalent to a comparison of two wrenches.

Equation (5.11) enables us readily to establish the comparison of two wrenches. Consider two wrenches which have the expressions, $\hat{w}_1=[\underline{f}_1;\ \underline{m}_{01}]$ and $\hat{w}_2=[\underline{f}_2;\ \underline{m}_{02}]$. These two wrenches are said to be equivalent if the conditions are satisfied: (1) $\underline{f}_1 = \underline{f}_2$, (2) $\underline{m}_{01} = \underline{m}_{02}$, which are as defined in Equation (5.10). Now, it is remained to compare two force vectors and two moment vectors. A vector is completely defined by its magnitude and direction cosine, which are geometrically invariant. Thus, the magnitudes and the direction cosines of two vectors are the two parameters considered to compare two vectors. The comparison of two vectors is established as following.

Theorem 5.1:Two wrenches are equivalent if the following conditions are satisfied,

(1) $\underline{f}_1 = \underline{f}_2$, then

(a) $|\underline{f}_1| = |\underline{f}_2|$, where $|\underline{f}_1|$ and $|\underline{f}_2|$ are the magnitude of the vectors $\underline{f}_1$, $\underline{f}_2$,

(b) $\cos^{-1}((\underline{f}_1/|\underline{f}_1|), \underline{f}_2/|\underline{f}_2|)) = 0$, where $((\underline{f}_1/|\underline{f}_1|),(\underline{f}_2/|\underline{f}_2|))$ is the expression of the inner product of two vectors.

(2) $\underline{m}_{01} = \underline{m}_{02}$, then

(a) $|\underline{m}_{01}| = |\underline{m}_{02}|$, where $|\underline{m}_{01}|$ and $|\underline{m}_{02}|$ are the magnitude of the vectors $\underline{m}_{01}$, $\underline{m}_{02}$

(b) $\cos^{-1}((\underline{m}_{01}/|\underline{m}_{01}|), \underline{m}_{02}/|\underline{m}_{02})) = 0$, where $((\underline{m}_{01}/|\underline{m}_{01}|),(\underline{m}_{02}/|\underline{m}_{02}|))$ is the expression of the inner product of two vectors.

The comparison based upon the physical meaning of the equivalent systems of forces is completely defined.

5.4 Comparison Using Equivalent Wrenches

In this section, we shall compare two wrench matrices to verify the experimental results. We had obtained two stiffness matrices, one is from a theoretical model, and one is from an experimental measurement. The two matrices have been shown in Sec.3.4 and are recalled here,

$$[K]_t = \begin{array}{rrrrrr} 23.34 & 0.00 & 0.00 & 10.93 & 84.27 & 0.00 \\ 0.00 & 23.34 & 0.00 & -84.27 & 10.93 & 0.00 \\ 0.00 & 0.00 & 70.80 & 0.00 & 0.00 & -21.86 \\ 10.93 & -84.27 & 0.00 & 489.43 & 0.00 & 0.00 \\ 84.27 & 10.93 & 0.00 & 0.00 & 489.43 & 0.00 \end{array}$$

$$\begin{matrix} 0.00 & 0.00 & -21.86 & 0.00 & 0.00 & 202.26 \end{matrix}$$

$$[K]_a = \begin{matrix} 24.02 & -0.61 & -0.26 & -23.21 & 93.37 & 2.67 \\ 1.10 & 24.74 & 1.69 & -84.30 & 17.07 & 0.56 \\ 5.51 & 1.42 & 70.07 & -6.94 & -23.13 & -21.87 \\ 20.10 & -82.88 & -15.99 & 563.11 & -61.18 & -8.64 \\ 70.05 & 0.88 & -1.53 & -26.65 & 552.63 & 2.26 \\ -7.20 & -3.99 & 0.60 & -19.02 & 28.71 & 241.87 \end{matrix}$$

where $[K]_t$ is the stiffness matrix obtained from kinematic model, and $[K]_a$ is the stiffness matrix from the actual measurement. The applied twist matrix $[\delta \underline{D}]_a$ in this case and actually measured wrench matrix $[\delta \underline{w}]_a$ are,

$$[\delta \underline{D}]_a = \begin{matrix} -0.00026 & -0.00809 & 0.03745 & -0.02218 & 0.00009 & 0.00000 \\ 0.05834 & -0.00490 & 0.00120 & -0.00193 & -0.02151 & 0.00005 \\ 0.00043 & -0.03906 & 0.00824 & -0.01704 & 0.00003 & 0.00000 \\ -0.00158 & 0.00001 & -0.00001 & -0.00000 & 0.00292 & -0.00001 \\ -0.00001 & 0.00000 & 0.00087 & -0.00299 & 0.00001 & 0.00000 \\ 0.00000 & 0.00000 & -0.00011 & 0.00036 & 0.00000 & -0.00326 \end{matrix}$$

$$[\delta \underline{w}]_a = \begin{matrix} -0.01 & -0.18 & 0.98 & -0.81 & -0.05 & -0.01 \\ 1.58 & -0.20 & 0.10 & -0.15 & -0.78 & 0.00 \\ 0.12 & -2.79 & 0.77 & -1.26 & -0.05 & 0.07 \\ -5.73 & 0.88 & 0.47 & 0.16 & 3.43 & 0.02 \\ 0.07 & -0.51 & 3.09 & -3.18 & -0.09 & -0.01 \\ -0.20 & 0.05 & -0.27 & 0.16 & 0.03 & -0.79 \end{matrix}$$

The experimental stiffness matrix $[K]_a$ is obtained from the composition of the wrench matrix, $[\delta \underline{w}]_a$, and the inverse of a twist matrix, $[\delta \underline{D}]_a^{-1}$. Then, we may calculate the theoretical wrench by using the theoretical stiffness matrix $[K]_t$ and the twist matrix $[\delta \underline{D}]_a$. From equation (5.9), the theoretical wrench matrix $[\delta w]_c$ due to the small twist, $[\delta \underline{D}]_a$, applied to the system, $[K]_t$, is,

$$[\delta w]_c = \begin{matrix} -0.02 & -0.19 & 0.95 & -0.77 & 0.03 & 0.00 \\ 1.49 & -0.12 & 0.04 & -0.08 & -0.75 & 0.00 \\ 0.03 & -2.77 & 0.59 & -1.21 & 0.00 & 0.07 \\ -5.69 & 0.33 & 0.30 & -0.08 & 3.24 & -0.01 \\ 0.61 & -0.74 & 3.59 & -3.35 & -0.22 & 0.00 \\ -0.01 & 0.85 & -0.20 & 0.45 & 0.00 & -0.66 \end{matrix}$$

Since the same twist is applied to the given systems, we may investigate the output wrenches of the systems for their agreement. The comparison is based upon the theorem defined in equation (5.11) in Sec.5.3 for equivalent systems of forces and it is carried out by computing the absolute difference of the magnitudes of two vectors, and the angle between two vectors following directly from Theorem 5.1. Let $\delta\underline{w}_{a,i}$ be denoted as the i-th column vector of the matrix $[\delta\underline{w}]_a$, where i=1, 2, 3,..,6. Since $\delta\underline{w}_{a,i} = [\delta\underline{f}_{a,i}; \delta\underline{m}_{a,i}]$, let $|\delta\underline{f}_{a,i}|$ denote the magnitude of $\delta\underline{f}_{a,i}$ and $|\delta\underline{m}_{a,i}|$ be the magnitude of $\delta\underline{m}_{a,i}$. Similarly, $\delta\underline{w}_{c,i}$ denotes the i-th column vector of the matrix $[\delta\underline{w}]_c$, where i=1 ,2 , 3,..,6; $\delta\underline{w}_{c,i} = [\delta\underline{f}_{c,i}; \delta\underline{m}_{c,i}]$, $|\delta\underline{f}_{c,i}|$ denotes the magnitude of $\delta\underline{f}_{c,i}$, and $|\delta\underline{m}_{c,i}|$ is the magnitude of $\delta\underline{m}_{c,i}$. The angle between two force vectors is given,

$$\gamma_{f,i} = \cos^{-1}\left(\frac{\underline{f}_{a,i}}{|\underline{f}_{a,i}|}, \frac{\underline{f}_{c,i}}{|\underline{f}_{c,i}|} \right), \tag{5.13}$$

where $\gamma_{f,i}$ is the angle between two force vectors. The angle between two moment vectors $\gamma_{m,i}$, where i=1,2,3..,6, is defined similarly to force vectors. This formulation will be not repeated here. Thus, we compare the actual measured system of forces and the computed system of forces. The error of the magnitude of the measured force vector of the computed force vector is defined by

$$\mu f_i = Abs(|\delta \underline{f}_{a,i}| - |\delta \underline{f}_{c,i}|). \tag{5.14}$$

The relative error of the magnitude of the measured force vector of the computed force vector is defined by

$$\mu f_{r,i} = \frac{\mu f_i}{|\delta \underline{f}_{c,i}|}. \tag{5.15}$$

The definition for the error of the magnitude of the measured moment vector of the computed moment vector, μm_i, is same to equation (5.14). The relative error of the magnitude of the measured moment vector of the computed moment vector, $\mu m_{r,i}$, is same to equation (5.15). Finally, the comparison results are shown in Table 5.1.

Table 5.1 The Comparison of Theoretical Wrenches Matrix and Experimental Wrench Matrix using Equivalent Wrenches

(1) force vectors:

[item]	$\lvert \delta \underline{f}_{a,i} \rvert$	$\lvert \delta \underline{f}_{c,i} \rvert$	μf_i	$\mu f_{r,i} \times 100\%$	$\gamma_{f,i}$ (degree)
i=1	1.58	1.49	0.09	6%	3.4°
i=2	2.80	2.78	0.02	1%	1.7°
i=3	1.25	1.12	0.13	11%	6.8°
i=4	1.51	1.44	0.07	4%	2.2°
i=5	0.78	0.75	0.03	4%	6.8°
i=6	0.07	0.07	0.00	0%	8.0°

(2) moment vectors

[item]	$\lvert \delta \underline{m}_{a,i} \rvert$	$\lvert \delta \underline{m}_{c,i} \rvert$	μm_i	$\mu m_{f,i} \times 100\%$	$\gamma_{m,i}$ (degree)
i=1	5.73	5.72	0.01	0%	5.7°
i=2	1.02	1.17	0.15	13%	53.2°
i=3	3.14	3.61	0.47	13%	4.0°
i=4	3.19	3.38	0.19	5%	6.3°
i=5	3.43	3.25	0.18	5%	2.3°
i=6	0.79	0.66	0.13	19%	2.3°

The results shown as above are considered better than the resolution expected. The results demonstrate the proximity of the two wrench matrices, which have similar wrenches. This means that two stiffness models produce similar physical effects when they are subjected to the same twist, so that the two stiffness models are related. In the comparison, the orientation of the second measured moment vector was misaligned with the computed moment vector in this case. This may happen when the components of one vector in a wrench are relatively small to the other vector in the same wrench and when the resolution is not sensitive enough. In this measure, a wrench vector is composed of two portions, one is a force vector, and the other is a moment vector. In the second column of the wrench matrix $[\delta \underline{w}]_c$, the moment vector is not the major part but the force vector is. An error incurred in the measurement had the components of the moment vector deviated.

As the result of the item 2 shown in the comparison of force vectors, the magnitude difference is Abs($\lvert \delta \underline{f}_{a,2} \rvert - \lvert \delta \underline{f}_{c,2} \rvert$)=0.02 (lb), and the angle $\gamma_{f,2}$ between the two force vectors, $\delta \underline{f}_{a,2}$ and $\delta \underline{f}_{c,2}$, is 1.7 (degree); the accuracy of the measurement is good.

However, the moment part (the 2nd item shown in the category of moment vectors) is the worst in this measurement. Overall, this measured wrench is still an admittable result when one considers the whole parts. The 1st and 5th measurements showed good examples of the accuracy for the measurement in this case. These evidences show the correlation between the two stiffness mappings, one is obtained from the experimental measurement, one is obtained from the theoretical model.

5.5 Comparison Using Geometric Invariants

In this section, we shall compare two wrench matrices using geometric invariants which has been introduced in Chapter 4. Two wrench matrices are given in preceding section. $[\delta\underline{w}]_a$ is an actual measured wrench changes due to the twist matrix $[\delta\underline{D}]_a$ applied to the compliant system. $[\delta w]_c$ is computed from the kinematic model when the same twists is supplied to the same system. For convenience, they are reproduced again,

$$[\delta\underline{w}]_a= \begin{matrix} -0.01 & -0.18 & 0.98 & -0.81 & -0.05 & -0.01 \\ 1.58 & -0.20 & 0.10 & -0.15 & -0.78 & 0.00 \\ 0.12 & -2.79 & 0.77 & -1.26 & -0.05 & 0.07 \\ -5.73 & 0.88 & 0.47 & 0.16 & 3.43 & 0.02 \\ 0.07 & -0.51 & 3.09 & -3.18 & -0.09 & -0.01 \\ -0.20 & 0.05 & -0.27 & 0.16 & 0.03 & -0.79 \end{matrix}$$

$$[\delta w]_c= \begin{matrix} -0.02 & -0.19 & 0.95 & -0.77 & 0.03 & -0.00 \\ 1.49 & -0.12 & 0.04 & -0.08 & -0.75 & 0.00 \\ 0.03 & -2.77 & 0.59 & -1.21 & 0.00 & 0.07 \\ -5.69 & 0.33 & 0.30 & -0.08 & 3.24 & -0.01 \\ 0.61 & -0.74 & 3.59 & -3.35 & -0.22 & 0.00 \\ -0.01 & 0.85 & -0.20 & 0.45 & -0.00 & -0.66 \end{matrix}$$

The comparison is carried out by comparing the magnitude difference, pitch difference, angle and perpendicular distance between two wrench screws from the wrench matrices $[\delta\underline{w}]_a$ and $[\delta\underline{w}]_c$. Let $\delta\underline{w}_{c,i}$ be denoted as the i-th column vector of the matrix $[\delta\underline{w}]_c$, where i=1,2,3,..,6, and $\delta\underline{w}_{a,i}$ be denoted as the i-th column vector of the matrix $[\delta\underline{w}]_c$, where i=1 ,2 , 3,..,6. The results for the comparison of two wrench screws, $\delta\underline{w}_{a,i}$ and $\delta\underline{w}_{c,i}$, where i=1 ,2 , 3,..,6, are shown in Table 5.2.

Table 5.2 A Comparison of Theoretical Wrench Matrix with Experimental Wrench Matrix Using Geometric Invarinats

[item]	$\delta \underline{w}_{a,1}$	$\delta \underline{w}_{c,1}$	absolute difference	R.E.
Magnitude	1.5846	1.4952	0.0894	6%
Pitch	0.0573	0.4698	0.4125	**
a_{12}	NM	NM	0.0829	
α_{12}	NM	NM	3.4(degree)	

[item]	$\delta \underline{w}_{a,2}$	$\delta \underline{w}_{c,2}$	absolute difference	R.E.
Magnitude	2.8029	2.7743	0.0286	1%
Pitch	-0.0249	-0.3038	0.2789	**
a_{12}	NM	NM	0.0665	
α_{12}	NM	NM	1.7(degree)	

[item]	$\delta \underline{w}_{a,3}$	$\delta \underline{w}_{c,3}$	absolute difference	R.E.
Magnitude	1.2503	1.1244	0.1259	11%
Pitch	0.3593	0.2469	0.1124	**
a_{12}	NM	NM	0.4043	
α_{12}	NM	NM	6.8(degree)	

[item]	$\delta \underline{w}_{a,4}$	$\delta \underline{w}_{c,4}$	absolute difference	R.E.
Magnitude	1.5054	1.4398	0.0656	4%
Pitch	0.0643	-0.1055	0.1698	**
a_{12}	NM	NM	0.2536	
α_{12}	NM	NM	2.2(degree)	

[item]	$\delta \underline{w}_{a,5}$	$\delta \underline{w}_{c,5}$	absolute difference	R.E.
Magnitude	0.7832	0.7488	0.0344	4%
Pitch	-0.1676	-0.4985	0.3309	**
a_{12}	NM	NM	0.0749	
α_{12}	NM	NM	6.8(degree)	

[item]	$\delta \underline{w}_{a,6}$	$\delta \underline{w}_{c,6}$	absolute difference	R.E.
Magnitude	0.0707	0.0713	0.0006	1%
Pitch	-11.1000	-9.2460	1.8540	**
a_{12}	NM	NM	1.2159	
α_{12}	NM	NM	8.0(degree)	

Note: 1.R.E: the "relative error" for this measurement of the computed wrench screw is defined similarly to Equation (5.15)
2. a_{12}: the perpendicular distance between two screws.
3. NM: No meaning for this category.
4. α_{12}: the angle between two screws

The quantities in these comparison categories demonstrate the agreement between two wrenches. The comparison results show that all geometric invariants are similar except the pitches. Since the measured forces and moments are subjected to the measurement errors, the locations of axes of the wrenches are changed so that it results in the pitches, the perpendicular distance and angle between the screws deviating.

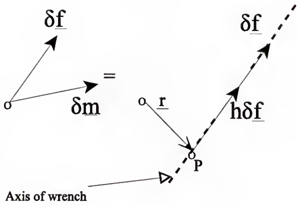

Figure 5.4 A wrench

The falling down on the evaluation of the pitches is the result of the error propogation from both the force and moment vectors. A wrench consists of a force and a couple, which is $\delta\underline{w}=[\delta\underline{f};\ \delta\underline{m}] = [\delta\underline{f};\ \underline{r}\times\delta\underline{f}\ ] + [\underline{0};\ h\delta\underline{f}]$, where h is the pitch of the wrench, and $\underline{r}$ is the position vector from the reference point to the line of action of $\delta\underline{f}$ (Figure 5.4). The pitch is obtained from

$$h=\frac{\delta\underline{f}\bullet\delta\underline{m}}{\delta\underline{f}\bullet\delta\underline{f}}. \tag{5.16}$$

The moment part of the wrench is resolved into a component $h\delta\underline{f}$ along $\delta\underline{f}$ and a component $\underline{r}\times\delta\underline{f}$ in a plane perpendicular to $\delta\underline{f}$. Thus, the error of the magnitude of the pitch is propagated from the ill-orientation and the magnitude of both of the force and moment vectors. As a result, the magnitude of the pitch is sensitive to the accuracy of the orientation and magnitude of the force and moment vectors in a wrench. Generally, if the difference of the magnitudes, the difference of the pitches, the perpendicular distance and angle between two screws are all within small numbers, the two wrenches are considered equivalent. However, there is no principal criterion to set an upper bound or lower bound to define the equivalence of the two pitches for two wrenches, neither for the absolute error of two pitches nor the relative error for two pitches. Thus, we can not determine the relatedness of two wrenches from the above results that three of the four quantities are similar but the pitches are not.

In this case, if we cannot determine the similarity of two wrenches by solely using the geometric invariants, we should apply the scheme mentioned in Sec.5.4. The definition of equivalent wrenches presented in Sec.5.4 are obtained from the generally accepted definition of the equivalent systems of forces in mechanics. From the comparison presented in Sec. 5.4, it has been known that the force vectors and moment vectors of the wrenches have approximate effects on the systems. Thus, the theoretical (kinematic) and actual models in such circumstance should be considered to have similar physical effects when the systems are subjected the same small twist.

As shown in the example, the comparison using geometric invariants examines the equivalence of two wrenches with a criterion which much criticizes on the accuracy of the measurement. This is the disadvantage of the scheme. However, the scheme using geometric invariants to compare the experimental wrench and theoretical wrenches is useful when the measurement have high resolution. This scheme also is good to for whom desires to inspect the individual effect of the push and the twist on the system by a wrench.

5.6 Comparison Using Potential Energy

We have verified the experimental result by using the concept of equivalent wrenches , geometric invariants in preceding sections. In this section, we introduce an alternative scheme to verify the stiffness matrices. This scheme is called "potential energy changing rate" . A small twist $\delta\underline{D}$ input to the compliant system induces a wrench change $\delta\underline{w}$ in the system. The reciprocal product (denoted as r.p) ,

$$r.p = \delta\underline{D}^T\delta\underline{w}, \tag{5.17}$$

which defines the virtual work done in the system. This formula is as defined in equation (2.11) in sec.2.2. The twist and wrench are related by a stiffness matrix [K], i.e. $\delta \underline{w} = [K]\delta \underline{D}$ (equation (2.1)). Substituting it into equation (5.17) and redenoting the equation as v.w obtains,

$$v.w = \delta \underline{D}^T [K] \delta \underline{D}. \tag{5.18}$$

The right-hand side of equation (5.18) is the quadric form of the second derivative of potential energy. Thus, equation (5.17) or (5.18) give the work done or the potential energy change in the system. These could be used to indicate the system characteristics.

For an actual model of a compliant system, the virtual work done by a twist to a compliant system can be calculated by equation (5.17)

$$v.w = \delta \underline{D}_{a,i}{}^T \delta \underline{w}_{a,i} \ , \tag{5.19}$$

where $\delta \underline{D}_{a,i}$ is the i-th column vectors of the actual measured twist matrix, and $\delta \underline{w}_{a,i}$. is the i-th column of the actual measured wrench matrix $[\delta \underline{w}]_a$. Similarly, for kinematic model of a complaint system, the virtual work done by a twist to a compliant system can be calculated by equation (5.17),

$$v.w = \delta \underline{D}_{a,i}{}^T \delta \underline{w}_{c,i}, \tag{5.20}$$

where $\delta \underline{D}_{a,i}$ is the same twist as in equation (5.19), and $\delta \underline{w}_{c,i}$. is the i-th column of the wrench matrix $[\delta \underline{w}]_c$ which is obtained from the theoretical calculation based upon the theoretical model.

In this investigation, $[\delta \underline{D}]_a$, $[\delta \underline{w}]_a$ and $[\delta \underline{w}]_c$ are the same matrices as appeared in the Sec. 5.4 and Sec.5.5. Using equation (5.19) and (5.20), the virtual work done by each twist and the corresponding wrench in the system is shown in Table 5.3.

Table 5.3 Comparison of Theoretical Wrenches and Experimental Wrenches Using Potential Energy Change Rate

[item]	$(\delta \underline{D}_{a,i})^T \delta \underline{w}_{a,i}$	$(\delta \underline{D}_{a,i})^T \delta \underline{w}_{c,i}$	R.E.x100%
i=1	0.101	0.096	4%
i=2	0.111	0.110	1%
i=3	0.046	0.044	4%
i=4	0.049	0.048	2%
i=5	0.027	0.026	4%
i=6	0.003	0.002	**

Note: R.E.= Abs($(\delta \underline{D}_{a,i})^T \delta \underline{w}_{a,i} - (\delta \underline{D}_{a,i})^T \delta \underline{w}_{c,i}$)/$(\delta \underline{D}_{a,i})^T \delta \underline{w}_{c,i}$

As the result shown in each categories of Table 5.3, the virtual works done by a twist with the wrenches reacted in the two systems are very close. This result represents that the same input twist to the systems produced the very close amount of virtual work done (or the change rate of the potential energy) in the systems. Thus, the theoretical (or kinematic) and the actual wrenches in such circumstance could be considered to generate the same potential energy change.

5.7 Summary

The verification of stiffness mappings is presented in the comparison of two wrench matrices, one is from theoretical calculation, and one is from the measurement. This comparison shown in Table 5.1 yields that the two models have similar wrench response. The absolute error is 0.13 (lb) in the force which corresponds to a relative error is 11%. The absolute error is 0.47(lb-in) which corresponds to a relative error 13%.

For a 6-DOF in-parallel compliant mechanism, the theoretical model of 6x6 stiffness matrices have been defined in equation (2.43). The experimental measurement of the stiffness matrix could be obtained as shown in equations (3.4) or (5.8) for the same mechanism. If there were no measurement errors in twist and wrench and modeling errors, the experimental stiffness matrix and theoretical stiffness matrix would be identical. An experimentally measured stiffness matrix in the project is limited to the apparatus resolution. The accuracy for generating an acceptable stiffness matrix requires extremely high resolution on determining twists and wrenches (ten millionth of an inch for twist and 10^{-3} lb for wrench measurement). This is because that the determination of stiffness matrices is very sensitive to the measurement errors. The characteristics of error propagation were examined in Sec.5.1.

Intuitively, the comparison could directly apply to the stiffness matrices themselves. Unfortunately, the comparison could not carry out because the measurement error sensitively propagates into the elements of the desired stiffness matrix. Under such a circumstance, the verification of stiffness mappings is in the insubstantial condition by comparing the elements of the stiffness matrix. Actually, the approach for comparing theoretical and experimental stiffness mappings could consider their wrench effects when they are subjected to the same twists. The definition shown in sec.5.2 establishes the foundation of physical meanings to quantify the characteristics of stiffness mappings. This definition leads us to further examine the two systems of wrenches occurring in the two models.

The equivalent systems of wrenches defined in Sec.5.3 characterize the physical effect of the given systems of force occurring in the two models. Thus, a comparison of two wrench matrices is established for investigating the relatedness of two stiffness mappings. The comparisons performed in Sec.5.4, Sec.5.5 and Sec.5.6 show that the two stiffness models have similar functions, i.e. they have the similar wrench effects for the same small twist. This comparison of the equivalent systems of forces demonstrates the relation between a small twist and wrench, which is expressed by a stiffness mapping. The physical model of a stiffness mapping can be established by a 6-DOF in-parallel compliant mechanism that provides the fulfillment of the theory of stiffness mapping in the theory and application.

5.8 Discussion

After those investigations done in the preceding sections, we should recall the experimental results shown in Sections 3.5 and 3.6. We have seen that both of two stiffness models are capable of being as parameters for 6-DOC wrench control in these cases shown in Sections 3.5 and 3.6. In addition to that, we see that two stiffness matrices don't have much likeness, but they produce alike wrench effects in the case which have been just investigated in this chapter. What is the significance that these experimental results have brought to us? We examined these results and a statement could be obtained: there exists a subspace, $\boldsymbol{W}_p = \boldsymbol{W}_{\mathrm{a}} \cap \boldsymbol{W}_{\mathrm{t}}$, which is an intersection of wrench spaces $\boldsymbol{W}_a$ and $\boldsymbol{W}_t$, such that $|\underline{a} - \underline{t}| < \underline{v}$, for $\underline{a} \in \boldsymbol{W}_a$, $\underline{t} \in \boldsymbol{W}_t$, where $\underline{v}$ is a small wrench vector whose elements are all positive numbers. Here, W_t is the theoretical wrench space, $\boldsymbol{W}_a$ is the experimental wrench space. This concept would be further interpreted as follows.

It has been known that a stiffness mapping, [K], relates a twist and a wrench by a stiffness matrix. It is true if the condition, $\delta\underline{D} \rightarrow \underline{0}$ but $\delta\underline{D} \neq \underline{0}$, is satisfied, where $\delta\underline{D}$ is a twist, and $\underline{0}$ is a 6x1 zero vector. A stiffness matrix [K] is obtained form the differential of the static force resultant equation. This follows the method established by Griffis and Duffy [3]. Practically, it is assumed that the theory is remained valid when the twist $\delta\underline{D}$ is bounded in a small interval, $\underline{\Omega} > \delta\underline{D} > \underline{0}$, which represents an open region of hyper-ellipsoid and zero is excluded. Thus, $[K]_t$ is the stiffness matrix for $\boldsymbol{S}_t$, which is a mapping $\boldsymbol{S}_t$: $\boldsymbol{D}_o \rightarrow \boldsymbol{W}_t$. Every wrench in the wrench space $\boldsymbol{W}_t$ is finite.

Under the assumption, $\underline{\Omega} > \delta\underline{D} > \underline{0}$, one could calculate the stiffness matrix via a set of six linearly independent twist vectors and a set of six linearly independent wrench vectors. The obtained $[K]_a$ is the stiffness matrix for $\boldsymbol{S}_a$, which is a mapping $\boldsymbol{S}_a$: $\boldsymbol{D}_o \rightarrow W_a$.

Thus, an experimental model and a theoretical stiffness model should converge while the condition, "$\delta\underline{D} \rightarrow \underline{0}$, but $\delta\underline{D} \neq \underline{0}$ ", is satisfied. Meanwhile, it is also assumed that there were no measurement errors and modeling errors have been involved. However, we may be unable to have an infinitesimal twist while we can measure a reasonable wrench and maintain appropriate stiffness for a physically modeled mechanism. We may be limited to the resolution of available measurement devices and facility in practical since the accuracy of [K] is extremely sensitive to the subjected measurement errors. Consequently, we may have a result that an experimental stiffness matrix wouldn't be alike to the theoretical stiffness matrix.

In this case, when two stiffness matrices don't have much likeness, it doesn't mean that they are not related. How can we know that they are "related"? We could examine the likelihood of their relatedness by their wrench effects. We have demonstrated the concept by comparing the theoretical wrench matrix and experimentally measured wrench matrix in the proceeding sections. The comparison has provided the evidence of relatedness of two wrench spaces. The relatedness between $\boldsymbol{W}_a$ and $\boldsymbol{W}_t$ is defined as that for $\underline{a} \in \boldsymbol{W}_a$, $\underline{t} \in \boldsymbol{W}_t$, $|\underline{a} - \underline{t}| < \underline{v}$, where $\underline{v}$ is a small wrench vector whose elements are all positive numbers. We should further present the definition to show that there exists a subspace $\boldsymbol{W}_p$ which is the intersection of the theoretical wrench space $\boldsymbol{W}_t$ and experimental wrench space $\boldsymbol{W}_a$.

The wrench space $\boldsymbol{W}_t$ is defined by a set of six linearly independent wrench vectors, i.e. $\boldsymbol{W}_t = \text{span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Here, the span of a set of vectors, $\text{span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, is defined as $\text{span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} = \{r_1\alpha_1 + r_2\alpha_2, + r_3\alpha_3 + r_4\alpha_4, + r_5\alpha_5 + r_6\alpha_6 \mid r_1, r_2, \ldots r_6$ are scalars$\}$. Assume that there is a wrench β_1, which is $\underline{\varepsilon}_1$ neighborhood of α_1, where $\underline{\varepsilon}_1$ is a relatively small wrench to α_1. Here, if $\underline{a}$ is any given wrench vector and ζ is any vector whose elements are all positive numbers (zero vector is excluded), we say that an ζ neighborhood of $\underline{a}$ is the set of all vectors that are less than ζ. We may define the other five wrenches, β_2, β_3, $\beta_{4,}$ β_5, and β_6, as β_1 in a similar manner. Here these six wrenches, β_1, β_2, β_3, $\beta_{4,}$ β_5, and β_6, are necessary to be linearly independent. The wrench space $\boldsymbol{W_a}$ defined by the set of six wrench vectors, i.e. $\boldsymbol{W_a} = \text{span}\{ \beta_1, \beta_2, \beta_3, \beta_{4,} \beta_5, \beta_6\}$. Because β_i is $\underline{\varepsilon}_i$ neighborhood of α_i, where i=1,2,..,6, the span$\{ \beta_1, \beta_2, \beta_3, \beta_{4,} \beta_5, \beta_6\}$ is a space which intersects with the span$\{\alpha_1,\alpha_2, \alpha_3, \alpha_{4,} \alpha_5, \alpha_6\}$ at a certain subspace $\boldsymbol{W_p}$, such that $\boldsymbol{W_p} = \boldsymbol{W_a} \cap \boldsymbol{W_t}$. Since two mappings are defined by, $\boldsymbol{S_t}$: $\boldsymbol{D_o} \rightarrow \boldsymbol{W_a}$, this intersection of two wrench spaces shows that the theoretical stiffness mapping, $\boldsymbol{S_t}$, and the experimental stiffness mapping, $\boldsymbol{S_a}$, are *related*.

Strictly speaking, it should not mean that the two mappings are equivalent in this investigation. It only could say that the two mappings are related but not equivalent. The experimentally measured stiffness mapping can transform a certain part of twist vectors in the twist space to the wrench space properly, but it cannot do so for the other part of twist vectors in this twist space. This is due to the mapping which is not the true mapping to transform every twist in the twist space to the corresponding wrench in the wrench space exactly. However, if we consider any arbitrary stiffness mapping, $\boldsymbol{S_x}$, which do not satisfy the assumption, then $\boldsymbol{S_x}$ would not be bale to transform any twist in the twist space $\boldsymbol{D_o}$ to the neighborhood of the wrench space $\boldsymbol{W_t}$ but $\boldsymbol{W_x}$ (Figure 5.3).

Consequently, a statement could be concluded from the overall investigations: Suppose that there are two stiffness mappings, $\boldsymbol{S_t}$ and $\boldsymbol{S_a}$. These mappings are given as, $\boldsymbol{S_t} : \boldsymbol{D_o} \rightarrow \boldsymbol{W_t}$, $\boldsymbol{S_a} : \boldsymbol{D_o} \rightarrow \boldsymbol{W_a}$. Two stiffness mappings are said "related", if there exists at least a twist $\underline{v}$ in twist space $\boldsymbol{D_o}$, i.e. $\underline{v} \in \boldsymbol{D_o}$, such that $|\boldsymbol{S_t}(\underline{v}) - \boldsymbol{S_a}(\underline{v})| < \underline{\varsigma}$ where $\underline{\varsigma}$ is a small wrench vector. Here, a stiffness mapping, for example, $\boldsymbol{S_a} : \boldsymbol{D_o} \rightarrow \boldsymbol{W_a}$, is denoted by $\boldsymbol{S_t}(\underline{v}) = [K]\underline{v}$ for all twist vectors $\underline{v} \in \boldsymbol{D_o}$, where [K] is a stiffness matrix for the mapping $\boldsymbol{S_a}$.

Now, recall the experimental results shown in sec.3.5 and sec.3.6. It was shown that a theoretical stiffness matrix and an experimental stiffness matrix were capable of controlling the force and position for a serial-chain robot manipulator. This is because in the three cases shown in Sec.3.6 the wrenches are happened in the subspace which is an intersection of the theoretical wrench space and an experimental wrench space. The *correlation* between two stiffness mappings discloses the experimental results shown in Sections 3.5 and 3.6.

It has been shown that the two wrench spaces have an intersection, and two stiffness mappings are "related" but not equivalent. The significance of the relatedness of the theoretical mapping and an experimental stiffness mapping is that the theoretical stiffness matrix exists and is an optimum parameter for being used to relate a twist in a twist space and a wrench in a wrench space.

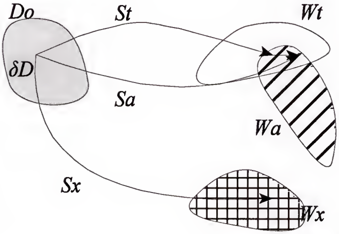

Figure 5.3 Mappings and their twist spaces, wrench spaces.

CHAPTER 6
SUMMARY, CONCLUSIONS AND FUTURE WORK

6.1 Summary and Conclusions

This study was an attempt to provide an experimental verification for a proposed stiffness model of a 6-DOF in-parallel compliant mechanism (6-DOF-IPCM). To achieve the study, a geometrically meaningful method to quantify the relation of a theoretically modeled stiffness matrix and an experimentally measured stiffness matrix is established. The issue considers Kinestatic Control which implements the theory of stiffness mapping to control the force and position for a robot manipulator.

A 6-DOF-IPCM has been modeled in this study as the physical device to realize the theory of stiffness mapping. Stiffness mapping is a geometric transformation which relates a small twist and a finite wrench by a 6x6 matrix. A stiffness mapping possesses many geometric properties; these properties have been discussed in Chapter 2 in order to understand a 6-DOF-IPCM.

Basically, the expression of a stiffness mapping can be categorized as two forms. If a stiffness mapping is a correlation, the mapping transforms an infinitesimal twist with axis coordinates into a wrench increment with ray coordinates. The correlation form of a stiffness mapping possesses the invariant property of the signature $[\pi,\upsilon,\zeta]$. If the mapping is a collineation, the mapping transforms screws from ray to ray coordinates (or axis to axis). The collineation has invariant properties including eigenstiffnesses and eigenscrews. Consequently, if the stiffness matrix is symmetric and positive-definite, the correlation form of a stiffness mapping has the signature [6,0,0], and the collineation form of the mapping has the signature [3,3,0]. The correlation form of a stiffness mapping can also be used to define the changing status of potential energy in the system. In this study, the eigenstiffnesses and eigenscrews are the invariant properties considered to explore the characteristics of a 6-DOF in-parallel compliant mechanism. A classification of stiffness mappings using this concept has been presented.

The existence of the wrench- and twist-compliant axes is another important characteristic of a stiffness mapping. The center of wrench-compliant axes and center of twist-compliant axes are the same if the stiffness mapping is symmetric. The concurrent point is called the center of elasticity. The eigen-properties of twist- and wrench-compliant are discussed. The classification of principal screws of TCA and WCA is given in Sec. 2.7. It is found that there is no principal screws of twist- and wrench-compliant axes having infinite pitch.

For a stiffness mapping, if there are two collinear eigenscrews whose eigenpitches are equal in magnitude and opposite in the sign, then a compliant axis exists for the compliant system. The condition results in the two corresponding eigenvalues with an equal magnitude and opposite sign; moreover, the two eigenscrews are reciprocal. The conclusion does not imply that the mapping is symmetric and positive-definite. This result is applied to any mapping of 6-DOF-IPCM. The proof has been shown in Sec.2.8.

In Chapter 3, a prototype of a 6-DOF-IPCM, which belongs to a class of special 6-6 Stewart-Platforms, has been developed. The 6-DOF-IPCM was implemented as a compliant device to model the stiffness of a series-chain robot manipulator. A stiffness matrix of the compliant system was experimentally determined via the measurement of a set of twist and wrench matrices. The stiffness matrix is a parameter to relate a small twist and a wrench for an in-parallel compliant mechanism. The theoretical model is used to testify the capability of stiffness mapping on the force and position control for a robot manipulator. This is accomplished by performing a 6-DOC wrench control, which have been presented in sec.3.5.

It has been shown that the theoretical model is feasible for controlling wrench increments by the application of the control law given in equation (3.5). The result indicates that a stiffness matrix exists for a compliant mechanism to relate a twist and a wrench. This means that Kinestatic Control for a robot manipulator can be accomplished by implementing a 6-DOF-IPCM.

The mathematic verifications of the stiffness matrices are also illustrated. There could be three different ways to verify the stiffness model. It can be carried out by comparing wrench matrices, twist matrices or stiffness matrices. These have been discussed in Chapters 4 and 5.

In Chapter 4, the number of geometric invariants involved in comparing a pair of geometrically like entities, such as points, planes, lines and screws, are enumerated. The method of comparing two given geometric entities is applied to active transformations of the plane and stiffness mappings of a 6-DOF-IPCM. This study demonstrates that there are no more than twenty-four independent parameters available for comparing one stiffness mapping with another.

The method of comparing two stiffness mappings using geometric invariants by comparing two stiffness matrices is carried out by finding their eigenstiffnesses and eigenscrews. This scheme has shown some advantages and disadvantages. The method is coordinate-free when it is applied to quantify the agreement between two mappings. The method classifies the properties of stiffness mappings and enumerate the number of the individual quantities involved. It has been found that there are four individual quantities, one four eigenstiffness differences, three for eigenscrews (i.e. angles between screws, perpendicular distances between screws, and pitch differences between screws). All of these quantities have distinct dimensions and are incompatible. No factor can be assigned to scale all of them to yield a unique quantity in the aspect of geometry. Therefore, the method cannot guarantee to yield an answer for comparing two stiffness mappings with a third.

In Chapter 5, the error propagation in the calculation of stiffness matrix is analyzed. The accuracy in the experimental determination of stiffness matrices is very sensitive to the measurement errors occurred in a twist matrix and a wrench matrix. It thus requires extremely high resolution in the measurement of twist and wrench matrices. The currently available facilities cannot meet the requirement to obtain an acceptable stiffness matrix. In view of this, we introduce the concept of equivalent wrenches. This concept can be further used to verify the stiffness models by the comparison of wrench matrices.

The definition of equivalent wrench matrices is derived from the observation of the wrench responses while a twist matrix is applied to the 6-DOF-IPCM. The concept is valid for the verification of the theoretical and experimental stiffness models for a 6-DOF-IPCM. This verification is carried out by comparing the two wrench matrices, one is from theoretical calculation, and the other is from experimental measurement. The comparison first utilizes the comparison model using the equivalent system of forces. Since the wrench matrix is composed of six wrench vectors as its column vectors, the comparison could directly compare a pair of column vectors as the comparison of a pair of force systems. The result shows that the two stiffness model produce close wrenches.

The comparison of wrench matrices using geometric invariants has been also demonstrated. The comparison model implemented here is based upon the comparison of a pair of screws. The result shows that the two stiffness models are related and close.

Another demonstration is intuitively derived from the reciprocal product of a twist and wrench screw. The reciprocal product represents the work input to system when a twist and a wrench incurred in the system. This quantity also represents the potential energy change when a twist applied to the system. The reciprocal products of six twist-and -wrench pairs for the two models were calculated. The result reveals that the change in potential energy due to a small twist applied to the system is almost the same.

Although the theoretical stiffness matrix and the experimental stiffness matrices are not really alike due to the limitation of resolution of measurement devices in this study, the two stiffness models show their similiarites between the wrench responses while a twist is applied. The verification of the wrench matrices shows that there exists an intersection between the two wrench spaces. This intersection is the subspace which can be used for the force and position control for a robot manipulator. The intersection of the theoretical wrench space and a experimental wrench space demonstrates the relatedness of the two stiffness mappings.

Strictly speaking, the existence of an intersection between two wrench spaces does not reveal the equivalence of two stiffness mappings. It only can be said that the theoretical stiffness mapping and experimental stiffness mapping are correlated. The significance of the correlation between the two stiffness mappings is the existence of the theoretical stiffness matrix. Theoretically, an experimental stiffness matrix would converge to the theoretical one if the condition, "$\delta \underline{D} \rightarrow \underline{0}$ but $\delta \underline{D} \neq 0$", were satisfied. The existence of the theoretical matrix demonstrates that the theoretical stiffness matrix is the optimum parameter to relate a twist and wrench at the position. The theoretical stiffness matrix represents the gradient of $\delta \underline{w}$ (wrench) respect to $\delta \underline{D}$ (twist) in the direction of steepest descent at $\delta \underline{w}$.

Technically, the exact stiffness matrix at the position may be able to be found experimentally if the resolutions of the apparatus are accurate enough, and there are no modeling errors involved. However, this usually means extremely high costs. If we are limited to the available facility, but we would like to have experimental stiffness matrices to achieve the desired needs. The verification of these matrices can be done by using the concept of equivalent wrench matrices to investigate the agreement between two models and set the bound of allowance.

6.2 Future Work

There may exist complex (or imaginary) eigenscrews in a real stiffness mapping. For a 6-DOF in-parallel compliant mechanism, its stiffness mapping always has six real eigenstiffnesses and six corresponding real eigenscrews when the compliant mechanism is at its nominal configuration. Imaginary eigenscrews may occur when a compliant mechanism is not at its nominal configuration. The eigenscrews are induced by complex (or imaginary) eigenstiffnesses. Dimentberg [14] mentioned in his literature, *The calculus and its Applications in Mechanics*, about two-systems of screws. He said that "*A linear combination of screws with real multipliers is a screw that lies on the same cylindroid as the basic screws; if the multipliers of the linear combination are complex, then the axis of the linear-combination screw describes a brush,* . . . " (p.140). In other words, two linearly independent screws with real multipliers form two-systems of screws which is a cylindroid. We may find that two real screws with complex multipliers actually determine a screw which is complex (or imaginary).

It is found that the correlation form of a stiffness mapping can have six real eigenvalues, which are not invariant under Euclidean transformation, but its collineation form may have complex eigenscrews, which are induced by complex eigenstiffnesses. What are the really physical meanings of imaginary eigenscrews for real stiffness mappings? The study of the invariant properties of stiffness mappings involves the characteristics of those complex and real eigenscrews. Refs.43, 44 and 45 might be helpful for this topic.

The trajectory of an in-parallel compliant mechanism should be continuous and smooth, but a singularity may occur. A singularity may occur when the determinant of

Jacobian matrix is zero. This singularity is a position singularity. For a stiffness mapping, a singularity occurs when the determinant of stiffness matrix is zero. This is different from the Jacobian matrix. These two singular conditions need to be considered when one models a 6-DOF-IPCM. Generally speaking, if one considers a mapping which is at the nominal position, a set of stiffness matrices can be generated when the mechanism moves around the position. The set of matrices is a set of matrices around the neighborhood of the nominal position. The investigation of this set of matrices are very helpful in understanding the behavior of 6-DOF-IPCM. The investigation on the set of matrices is suggested using Catastrophe theory. References 46, 47 and 48 might be helpful for this topic. The approach is to consider any infinitesimal wrench (or infinitesimal twist) applied to the system as a perturbation to the system. The analysis of stiffness matrices reveals detailed information about how the characteristics change when the system changes its configuration. The information may benefit on the topics of Kinestatic Control and task planning for robot manipulators.

The theory of stiffness mapping has been experimentally testified to be capable of controlling the wrench increment for a robot manipulator in this study. The development of the next generation of a 6-DOF in-parallel mechanism may be useful in the industrial applications.

APPENDIX A
LIST OF EXPERIMENTAL DATA

The following is a list of the experimental data which are generated from the measurement of twist and wrench screws. A twist is represented in axial coordinates, i.e. $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$. The two 3x1 vectors of the twist are the small translation $\delta\underline{x}$ and the small rotation angle $\delta\underline{\theta}$. A wrench is represented in the ray coordinates, i.e. $\delta\underline{w}=[\delta\underline{f}\,;\delta\underline{m}_0]$. The two 3x1 vectors of the wrench are small force change $\delta\underline{f}$ and small moment change $\delta\underline{m}_0$.

In the experiment, the parameters of the GE-P60 robot varied from experiment to experiment (see details in Sec.3.3).

In the twist category, each data block represents a set of data. The set of data are the measured twist changes when the same command was given μ times during the experiment. Each recorded twist is represented in a row at each set of data and is read from left to right. The first three components form a translation vector, and the latter three components form a rotation vector.

In the wrench category, each data block represents a set of data. The set of data are the wrench changes to the corresponding twists. A wrench change is measured when a twist change was applied to the system during the experiment. A measured wrench is represented in a row at each set of data and is read from left to right. The first three components form a force vector, and the latter three components form a moment vector.

$[\delta\underline{D}]_a$ represents the generated twist matrix from the experiment. Each entity of the twist matrix is a mean value of these twist components. $[\delta\underline{w}]_a$ represents the generated wrench matrix from experiment. Each column of the wrench matrix is a mean value of these wrench vectors, which is the average of the sum total of measured wrenches for the corresponding commanded twists. [Ka] is the stiffness matrix computed by directly following using equation (3.4), i.e. $[Ka]= [\delta\underline{w}]_a[\delta\underline{D}]_a^{-1}$. For cross reference, $[\delta\underline{w}]_c$ is calculated from the equation, $[\delta\underline{w}]_c= [K]_t\,[\delta\underline{D}]_a$, where $[K]_t$ is theoretical stiffness matrix. Here, $[\delta\underline{w}]_c$ is defined as the theoretical wrench matrix. The experiment data are listed thereafter.

(1) Twist screw, $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$.

```
 Entity
  (1,1)     (2,1)     (3,1)     (4,1)     (5,1)     (6,1)
-0.00026  0.05856  0.00043 -0.00157 -0.00001 -0.00000
-0.00026  0.05769  0.00042 -0.00159 -0.00001 -0.00000
-0.00026  0.05891  0.00043 -0.00162 -0.00001 -0.00000
-0.00026  0.05856  0.00043 -0.00157 -0.00001 -0.00000
-0.00026  0.05734  0.00042 -0.00154 -0.00001 -0.00000
-0.00026  0.05856  0.00043 -0.00157 -0.00001 -0.00000
-0.00026  0.05891  0.00043 -0.00162 -0.00001 -0.00000
-0.00026  0.05821  0.00043 -0.00152 -0.00001 -0.00000
 Entity
  (1,2)     (2,2)     (3,2)     (4,2)     (5,2)     (6,2)
-0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
-0.00809 -0.00479 -0.03906  0.00000  0.00000  0.00000
-0.00809 -0.00515 -0.03906  0.00005  0.00000  0.00000
-0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
-0.00809 -0.00515 -0.03906  0.00005  0.00000  0.00000
-0.00809 -0.00479 -0.03906  0.00000  0.00000  0.00000
-0.00809 -0.00479 -0.03906  0.00000  0.00000  0.00000
 Entity
  (1,3)     (2,3)     (3,3)     (4,3)     (5,3)     (6,3)
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03744  0.00151  0.00824 -0.00005  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03744  0.00151  0.00824 -0.00005  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03745  0.00081  0.00824  0.00004  0.00087 -0.00011
 Entity
  (1,4)     (2,4)     (3,4)     (4,4)     (5,4)     (6,4)
-0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
-0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
-0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
-0.02222 -0.00095 -0.01705 -0.00002 -0.00299  0.00036
-0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
-0.02221 -0.00182 -0.01706 -0.00003 -0.00299  0.00036
-0.02198 -0.00180 -0.01688 -0.00003 -0.00296  0.00036
-0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
 Entity
  (1,5)     (2,5)     (3,5)     (4,5)     (5,5)     (6,5)
 0.00009 -0.02166  0.00003  0.00292  0.00001 -0.00000
 0.00009 -0.02079  0.00004  0.00294  0.00001 -0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001 -0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001 -0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001 -0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001 -0.00000
```

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00320
-0.00000	0.00035	0.00000	-0.00005	-0.00000	-0.00327
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
-0.00000	0.00035	0.00000	-0.00005	-0.00000	-0.00327

Wrench screw, $\delta\underline{w}=[\delta\underline{f}; \delta\underline{m}_0]$

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.01	1.56	0.11	-5.74	0.06	-0.16
-0.01	1.59	0.12	-5.74	0.06	-0.20
-0.01	1.60	0.12	-5.75	0.09	-0.20
0.00	1.58	0.12	-5.75	0.09	-0.18
-0.02	1.58	0.12	-5.74	0.06	-0.21
0.00	1.57	0.13	-5.70	0.06	-0.22
0.00	1.58	0.13	-5.75	0.08	-0.22
0.00	1.55	0.13	-5.70	0.05	-0.22

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.17	-0.19	-2.76	0.87	-0.49	0.05
-0.17	-0.18	-2.74	0.88	-0.47	0.05
-0.17	-0.19	-2.77	0.89	-0.48	0.05
-0.17	-0.19	-2.80	0.87	-0.49	0.05
-0.20	-0.21	-2.82	0.87	-0.56	0.06
-0.19	-0.21	-2.82	0.89	-0.53	0.06
-0.20	-0.21	-2.81	0.86	-0.56	0.06

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.98	0.10	0.77	0.49	3.19	-0.27
0.99	0.11	0.78	0.46	3.18	-0.28
0.99	0.10	0.76	0.48	3.18	-0.28
0.98	0.11	0.77	0.46	3.08	-0.28
0.98	0.10	0.77	0.47	3.10	-0.29
0.97	0.10	0.76	0.45	3.05	-0.25
0.97	0.09	0.77	0.46	3.06	-0.26
0.97	0.10	0.75	0.46	2.98	-0.27
0.97	0.09	0.78	0.46	3.01	-0.26

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.80	-0.15	-1.28	0.16	-3.21	0.14
-0.80	-0.15	-1.30	0.16	-3.22	0.13
-0.81	-0.15	-1.20	0.17	-3.18	0.16
-0.81	-0.16	-1.20	0.17	-3.13	0.18
-0.79	-0.15	-1.29	0.17	-3.13	0.17
-0.81	-0.15	-1.29	0.17	-3.14	0.18
-0.81	-0.15	-1.21	0.16	-3.20	0.16
-0.81	-0.15	-1.29	0.15	-3.22	0.15

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.05	-0.77	-0.02	3.36	-0.06	0.00
-0.04	-0.78	-0.06	3.46	-0.09	0.04
-0.04	-0.75	-0.05	3.32	-0.08	0.04
-0.06	-0.80	-0.06	3.60	-0.10	0.03
-0.06	-0.75	-0.05	3.32	-0.08	0.03
-0.06	-0.82	-0.05	3.52	-0.10	0.04

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

```
-0.01   -0.00    0.08    0.02   -0.01   -0.80
-0.01   -0.00    0.08    0.02   -0.01   -0.79
-0.00    0.00    0.07    0.02    0.00   -0.76
-0.01    0.00    0.07    0.02   -0.01   -0.80
-0.01   -0.00    0.08    0.02   -0.01   -0.79
-0.01    0.00    0.06    0.02   -0.01   -0.79
-0.01    0.00    0.06    0.02    0.00   -0.79
```

$[\delta\underline{D}]_a$=

```
-0.00026  -0.00809   0.03745  -0.02218   0.00009   0.00000
 0.05834  -0.00490   0.00120  -0.00193  -0.02151   0.00005
 0.00043  -0.03906   0.00824  -0.01704   0.00003   0.00000
-0.00158   0.00001  -0.00001  -0.00000   0.00292  -0.00001
-0.00001   0.00000   0.00087  -0.00299   0.00001   0.00000
 0.00000   0.00000  -0.00011   0.00036   0.00000  -0.00326
```

$[\delta\underline{w}]_a$=

```
-0.01   -0.18    0.98   -0.81   -0.05   -0.01
 1.58   -0.20    0.10   -0.15   -0.78    0.00
 0.12   -2.79    0.77   -1.26   -0.05    0.07
-5.73    0.88    0.47    0.16    3.43    0.02
 0.07   -0.51    3.09   -3.18   -0.09   -0.01
-0.20    0.05   -0.27    0.16    0.03   -0.79
```

[Ka]=

```
 24.02   -0.61   -0.26   -23.21    93.37     2.67
  1.10   24.74    1.69   -84.30    17.07     0.56
  5.51    1.42   70.07    -6.94   -23.13   -21.87
 20.10  -82.88  -15.99   563.11   -61.18    -8.64
 70.05    0.88   -1.53   -26.65   552.63     2.26
 -7.20   -3.99    0.60   -19.02    28.71   241.87
```

$[\delta\underline{w}]_c$=

```
-0.02   -0.19    0.95   -0.77    0.03    0.00
 1.49   -0.12    0.04   -0.08   -0.75    0.00
 0.03   -2.77    0.59   -1.21    0.00    0.07
-5.69    0.33    0.30   -0.08    3.24   -0.01
 0.61   -0.74    3.59   -3.35   -0.22    0.00
-0.01    0.85   -0.20    0.45    0.00   -0.66
```

(2)

Twist screw, $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.00026	0.05734	0.00042	-0.00154	-0.00001	-0.00000
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000
-0.00026	0.05769	0.00042	-0.00159	-0.00001	-0.00000
-0.00025	0.05647	0.00041	-0.00155	-0.00001	-0.00000
-0.00026	0.05734	0.00042	-0.00154	-0.00001	-0.00000
-0.00025	0.05577	0.00041	-0.00146	-0.00001	-0.00000

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000
-0.00832	-0.00517	-0.03924	0.00005	-0.00003	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000
-0.00843	-0.00488	-0.03973	0.00000	-0.00003	0.00000

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03722	0.00114	0.00807	-0.00000	0.00084	-0.00010
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011

```
 0.03744  0.00151  0.00824 -0.00005  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
 0.03722  0.00079  0.00807  0.00004  0.00084 -0.00010
 0.03721  0.00149  0.00807 -0.00005  0.00084 -0.00010
 0.03744  0.00151  0.00824 -0.00005  0.00087 -0.00011
 Entity
  (1,4)    (2,4)    (3,4)    (4,4)    (5,4)    (6,4)
-0.02244 -0.00254 -0.01723  0.00006 -0.00302  0.00037
-0.02143 -0.00215 -0.01689  0.00001 -0.00298  0.00036
-0.02299 -0.00255 -0.01723  0.00006 -0.00301  0.00037
-0.02244 -0.00220 -0.01723  0.00001 -0.00302  0.00037
-0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
-0.02221 -0.00252 -0.01706  0.00006 -0.00299  0.00036
-0.02244 -0.00220 -0.01723  0.00001 -0.00302  0.00037
-0.02299 -0.00220 -0.01723  0.00001 -0.00301  0.00037
-0.02221 -0.00252 -0.01706  0.00006 -0.00299  0.00036
 Entity
  (1,5)    (2,5)    (3,5)    (4,5)    (5,5)    (6,5)
 0.00009 -0.02201  0.00003  0.00297  0.00001  0.00000
 0.00009 -0.02131  0.00003  0.00287  0.00001  0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
 0.00009 -0.02201  0.00003  0.00297  0.00001  0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
 0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
 Entity
  (1,6)    (2,6)    (3,6)    (4,6)    (5,6)    (6,6)
-0.00000 -0.00035  0.00000  0.00005  0.00000 -0.00327
 0.00000 -0.00035  0.00000 -0.00005 -0.00000 -0.00327
 0.00000 -0.00035  0.00000  0.00005  0.00000 -0.00320
 0.00000 -0.00035  0.00000  0.00005  0.00000 -0.00327
 0.00000 -0.00035  0.00000  0.00005  0.00000 -0.00327
 0.00000 -0.00035  0.00000  0.00005  0.00000 -0.00327
```

Wrench screw, $\delta\underline{w}=[\delta\underline{f};\ \delta\underline{m}_0]$

```
 Entity
 (1,1)   (2,1)  (3,1)  (4,1)   (5,1)  (6,1)
 -0.09   1.50   0.04   -5.58   0.10   -0.04
 -0.09   1.54   0.05   -5.72   0.10   -0.02
 -0.11   1.51   0.05   -5.78   0.12   -0.02
 -0.12   1.55   0.04   -5.60   0.10   -0.03
 -0.11   1.56   0.04   -5.66   0.11   -0.06
 -0.14   1.55   0.04   -5.66   0.06   -0.06
 -0.12   1.56   0.05   -5.61   0.09   -0.06
 Entity
 (1,2)   (2,2)  (3,2)  (4,2)   (5,2)  (6,2)
 -0.17   -0.16  -2.81  0.95    -0.41  0.07
 -0.17   -0.15  -2.82  0.94    -0.30  0.06
 -0.15   -0.15  -2.76  0.99    -0.33  0.07
 -0.15   -0.16  -2.78  0.98    -0.31  0.06
 -0.16   -0.15  -2.78  0.99    -0.29  0.06
 -0.15   -0.15  -2.77  0.97    -0.29  0.06
 -0.16   -0.15  -2.80  0.98    -0.33  0.07
 -0.14   -0.15  -2.78  0.98    -0.31  0.07
 Entity
 (1,3)   (2,3)  (3,3)  (4,3)   (5,3)  (6,3)
 1.01    0.05   0.66   0.52    3.05   -0.18
 1.01    0.05   0.70   0.53    3.08   -0.20
 1.00    0.05   0.61   0.53    3.07   -0.18
 1.02    0.04   0.65   0.51    3.08   -0.17
 1.02    0.04   0.68   0.53    3.01   -0.20
```

1.01	0.05	0.69	0.51	3.00	-0.19
1.00	0.04	0.62	0.54	3.03	-0.18
1.01	0.05	0.71	0.58	3.05	-0.24
1.01	0.06	0.66	0.53	3.02	-0.22

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.84	-0.11	-1.22	0.10	-3.12	0.18
-0.82	-0.12	-1.21	0.10	-3.13	0.18
-0.82	-0.11	-1.20	0.10	-3.09	0.21
-0.82	-0.11	-1.20	0.10	-3.11	0.22
-0.82	-0.11	-1.21	0.10	-3.11	0.17
-0.82	-0.11	-1.22	0.10	-3.12	0.20
-0.82	-0.11	-1.21	0.10	-3.11	0.20
-0.81	-0.11	-1.21	0.10	-3.11	0.21
-0.81	-0.11	-1.19	0.10	-3.11	0.20

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.05	-0.76	-0.02	3.48	-0.03	-0.01
-0.05	-0.75	-0.02	3.45	-0.10	0.03
-0.05	-0.76	-0.02	3.44	-0.09	0.02
-0.05	-0.77	-0.02	3.47	-0.13	0.03
-0.05	-0.76	-0.02	3.46	-0.11	0.03
-0.05	-0.76	-0.02	3.46	-0.13	0.05

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.02	0.00	0.07	-0.02	-0.02	-0.76
-0.00	-0.00	0.06	0.00	0.01	-0.75
-0.01	0.01	0.06	-0.01	0.01	-0.78
-0.01	0.01	0.07	-0.01	0.01	-0.77
-0.01	0.01	0.09	-0.01	0.01	-0.76
-0.01	0.01	0.08	-0.01	0.01	-0.75

$$[\delta\underline{D}]_a=\begin{array}{rrrrrr}
-0.00025 & -0.00812 & 0.03737 & -0.02237 & 0.00009 & 0.00000 \\
0.05694 & -0.00488 & 0.00123 & -0.00234 & -0.02172 & -0.00035 \\
0.00042 & -0.03908 & 0.00818 & -0.01714 & 0.00003 & 0.00000 \\
-0.00152 & 0.00001 & -0.00001 & 0.00003 & 0.00293 & 0.00003 \\
-0.00001 & -0.00000 & 0.00086 & -0.00300 & 0.00001 & 0.00000 \\
0.00000 & 0.00000 & -0.00011 & 0.00037 & 0.00000 & -0.00326
\end{array}$$

$$[\delta\underline{w}]_a=\begin{array}{rrrrrr}
-0.11 & -0.17 & 1.01 & -0.82 & -0.05 & -0.01 \\
1.54 & -0.16 & 0.05 & -0.11 & -0.76 & 0.01 \\
0.04 & -2.80 & 0.66 & -1.21 & -0.02 & 0.07 \\
-5.66 & 0.97 & 0.53 & 0.10 & 3.46 & -0.01 \\
0.10 & -0.37 & 3.04 & -3.11 & -0.10 & 0.01 \\
-0.04 & 0.07 & -0.20 & 0.20 & 0.02 & -0.76
\end{array}$$

$$[Ka]=\begin{array}{rrrrrr}
25.15 & -2.88 & -0.65 & -39.52 & 91.53 & 2.97 \\
-0.05 & 25.04 & 0.99 & -73.87 & 10.79 & -5.49 \\
2.78 & 0.08 & 70.97 & -6.96 & -26.30 & -22.07 \\
21.62 & -84.27 & -18.52 & 556.12 & -14.94 & 17.81 \\
70.04 & 1.43 & -5.34 & -26.90 & 543.30 & -1.96 \\
-4.38 & -0.63 & -0.83 & 3.97 & 0.90 & 233.87
\end{array}$$

$$[\delta\underline{w}]_c=\begin{array}{rrrrrr}
-0.02 & -0.19 & 0.94 & -0.77 & 0.03 & 0.00 \\
1.46 & -0.11 & 0.04 & -0.09 & -0.75 & -0.01 \\
0.03 & -2.77 & 0.58 & -1.22 & 0.00 & 0.07
\end{array}$$

```
 -5.55   0.33   0.30  -0.03   3.26   0.05
  0.60  -0.74   3.58  -3.38  -0.22  -0.00
 -0.01   0.85  -0.20   0.45   0.00  -0.66
```

(3)
Twist screw, $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$.

```
   Entity
    (1,1)     (2,1)     (3,1)     (4,1)     (5,1)     (6,1)
 -0.00025  0.05612  0.00041 -0.00151 -0.00001 -0.00000
 -0.00025  0.05577  0.00041 -0.00146 -0.00001 -0.00000
 -0.00026  0.05769  0.00042 -0.00159 -0.00001 -0.00000
 -0.00026  0.05769  0.00042 -0.00159 -0.00001 -0.00000
 -0.00026  0.05769  0.00042 -0.00159 -0.00001 -0.00000
 -0.00025  0.05612  0.00041 -0.00151 -0.00001 -0.00000
 -0.00025  0.05699  0.00042 -0.00149 -0.00001 -0.00000
   Entity
    (1,2)     (2,2)     (3,2)     (4,2)     (5,2)     (6,2)
 -0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
 -0.00833 -0.00482 -0.03924  0.00000 -0.00003  0.00000
 -0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
 -0.00833 -0.00482 -0.03924  0.00000 -0.00003  0.00000
 -0.00832 -0.00482 -0.03924  0.00000 -0.00003 -0.00006
 -0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
 -0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
 -0.00809 -0.00480 -0.03906  0.00000  0.00000  0.00000
   Entity
    (1,3)     (2,3)     (3,3)     (4,3)     (5,3)     (6,3)
  0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
  0.03644  0.00112  0.00789 -0.00000  0.00082 -0.00010
  0.03755  0.00157  0.00873 -0.00005  0.00087 -0.00011
  0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
  0.03744  0.00151  0.00824 -0.00005  0.00087 -0.00011
  0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
  0.03745  0.00116  0.00824 -0.00000  0.00087 -0.00011
   Entity
    (1,4)     (2,4)     (3,4)     (4,4)     (5,4)     (6,4)
 -0.02221 -0.00252 -0.01706  0.00006 -0.00299  0.00043
 -0.02244 -0.00254 -0.01723  0.00006 -0.00302  0.00043
 -0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
 -0.02221 -0.00252 -0.01706  0.00006 -0.00299  0.00036
 -0.02221 -0.00217 -0.01706  0.00001 -0.00299  0.00036
 -0.02244 -0.00254 -0.01723  0.00006 -0.00302  0.00037
 -0.02198 -0.00215 -0.01688  0.00001 -0.00296  0.00036
 -0.02244 -0.00220 -0.01723  0.00001 -0.00302  0.00037
 -0.02244 -0.00219 -0.01724  0.00001 -0.00302  0.00037
 -0.02221 -0.00252 -0.01706  0.00006 -0.00299  0.00036
   Entity
    (1,5)     (2,5)     (3,5)     (4,5)     (5,5)     (6,5)
  0.00009 -0.02201  0.00003  0.00297  0.00001  0.00000
  0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
  0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
  0.00009 -0.02096  0.00003  0.00283  0.00001  0.00000
  0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
  0.00009 -0.02131  0.00003  0.00287  0.00001  0.00000
  0.00009 -0.02201  0.00003  0.00297  0.00001  0.00000
  0.00009 -0.02166  0.00003  0.00292  0.00001  0.00000
   Entity
    (1,6)     (2,6)     (3,6)     (4,6)     (5,6)     (6,6)
  0.00000  0.00000  0.00000  0.00000 -0.00000 -0.00314
  0.00000  0.00000  0.00000  0.00000 -0.00000 -0.00327
  0.00000  0.00000  0.00000  0.00000  0.00000 -0.00320
 0.00000  0.00000  0.00000  0.00000 -0.00000 -0.00314
 0.00000  0.00000  0.00000 -0.00000  0.00000 -0.00327
-0.00000  0.00035  0.00000 -0.00005 -0.00000 -0.00320
-0.00000  0.00035  0.00000 -0.00005 -0.00000 -0.00327
 0.00000  0.00000  0.00000 -0.00000  0.00000 -0.00327
 0.00000  0.00000  0.00000  0.00000 -0.00000 -0.00314
 0.00000  0.00000  0.00000  0.00000 -0.00000 -0.00320
```

Wrench screw, $\delta\underline{w}=[\delta\underline{f}; \delta\underline{m}_0]$

```
 Entity
 (1,1)   (2,1)   (3,1)   (4,1)   (5,1)   (6,1)
 -0.05   1.61    0.08   -5.55    0.04   -0.14
 -0.05   1.56    0.09   -5.56    0.01   -0.12
 -0.05   1.57    0.09   -5.59    0.01   -0.14
 -0.03   1.56    0.08   -5.60    0.01   -0.14
 -0.05   1.54    0.09   -5.65   -0.00   -0.12
 -0.04   1.54    0.10   -5.66    0.01   -0.13
 -0.03   1.54    0.10   -5.67    0.03   -0.14
 Entity
 (1,2)   (2,2)   (3,2)   (4,2)   (5,2)   (6,2)
 -0.15  -0.18   -2.78    0.81   -0.29    0.05
 -0.15  -0.18   -2.81    0.80   -0.32    0.06
 -0.15  -0.19   -2.76    0.85   -0.26    0.06
 -0.15  -0.19   -2.81    0.89   -0.29    0.05
 -0.15  -0.19   -2.80    0.86   -0.29    0.06
 -0.14  -0.19   -2.81    0.89   -0.23    0.06
 -0.16  -0.21   -2.83    0.89   -0.27    0.05
 -0.16  -0.20   -2.80    0.90   -0.28    0.05
 Entity
 (1,3)   (2,3)   (3,3)   (4,3)   (5,3)   (6,3)
 1.02    0.09    0.71    0.49    3.00   -0.23
 1.01    0.11    0.70    0.54    2.96   -0.23
 1.02    0.11    0.71    0.51    2.97   -0.21
 1.02    0.09    0.71    0.54    2.95   -0.20
 1.05    0.09    0.71    0.50    2.95   -0.23
 1.02    0.09    0.72    0.52    2.96   -0.20
 1.00    0.10    0.71    0.48    2.71   -0.18
 Entity
 (1,4)   (2,4)   (3,4)   (4,4)   (5,4)   (6,4)
 -0.86  -0.19   -1.26    0.08   -3.01    0.15
 -0.82  -0.18   -1.19    0.08   -3.00    0.21
 -0.82  -0.18   -1.20    0.08   -3.01    0.23
 -0.82  -0.18   -1.22    0.08   -3.00    0.25
 -0.82  -0.20   -1.23    0.08   -2.99    0.24
 -0.85  -0.19   -1.24    0.08   -3.03    0.25
 -0.84  -0.19   -1.20    0.07   -3.02    0.25
 -0.84  -0.18   -1.25    0.08   -3.00    0.24
 -0.85  -0.19   -1.25    0.08   -3.02    0.25
 -0.86  -0.19   -1.26    0.08   -3.01    0.24
 Entity
 (1,5)   (2,5)   (3,5)   (4,5)   (5,5)   (6,5)
 -0.06  -0.77    0.00    3.45   -0.03   -0.00
 -0.07  -0.76   -0.04    3.26   -0.09    0.02
 -0.07  -0.81   -0.07    3.46   -0.10    0.01
 -0.07  -0.79   -0.05    3.44   -0.11    0.01
 -0.06  -0.77   -0.06    3.44   -0.11    0.03
 -0.06  -0.79   -0.07    3.45   -0.12    0.03
 -0.07  -0.75   -0.08    3.43   -0.10    0.06
 -0.07  -0.80   -0.08    3.40   -0.10    0.06
 Entity
 (1,6)   (2,6)   (3,6)   (4,6)   (5,6)   (6,6)
```

```
-0.02   0.00   0.06  -0.02  -0.01  -0.71
-0.01  -0.01   0.07   0.01   0.01  -0.73
-0.01   0.00   0.07  -0.00   0.00  -0.73
-0.01   0.01   0.06  -0.00   0.01  -0.74
-0.01   0.01   0.06  -0.01   0.01  -0.73
-0.01   0.01   0.06  -0.00   0.01  -0.70
-0.01   0.01   0.07  -0.00   0.01  -0.70
-0.01   0.01   0.06  -0.01   0.01  -0.74
-0.01   0.01   0.07  -0.00   0.01  -0.73
-0.02   0.01   0.07  -0.00   0.00  -0.74
```

$[\delta\underline{D}]_a=$

```
-0.00025  -0.00818   0.03732  -0.02228   0.00009   0.00000
 0.05687  -0.00481   0.00126  -0.00235  -0.02162   0.00007
 0.00042  -0.03913   0.00826  -0.01711   0.00003   0.00000
-0.00153   0.00000  -0.00001   0.00004   0.00291  -0.00001
-0.00001  -0.00001   0.00086  -0.00300   0.00001   0.00000
 0.00000  -0.00001  -0.00011   0.00038   0.00000  -0.00321
```

$[\delta\underline{w}]_a=$

```
-0.04  -0.15   1.02  -0.84  -0.07  -0.01
 1.56  -0.19   0.10  -0.19  -0.78   0.01
 0.09  -2.80   0.71  -1.23  -0.06   0.07
-5.61   0.86   0.51   0.08   3.42  -0.00
 0.02  -0.28   2.93  -3.01  -0.10   0.01
-0.13   0.05  -0.21   0.23   0.03  -0.72
```

[Ka]=

```
25.36   -1.57   -1.27  -35.49   99.57    3.81
 0.74   25.26    1.62  -80.40   26.74   -1.07
 3.88    0.67   70.68  -15.07  -25.16  -20.19
20.68  -83.58  -16.06  551.72  -16.66   -2.61
67.51   -0.32   -7.10  -38.85  541.94   -1.76
-4.62   -2.62   -0.16   -9.83  -11.45  225.83
```

$[\delta\underline{w}]_c=$

```
-0.02  -0.19   0.94  -0.77   0.03   0.00
 1.46  -0.11   0.04  -0.09  -0.75   0.00
 0.03  -2.77   0.59  -1.22   0.00   0.07
-5.55   0.32   0.29  -0.03   3.25  -0.01
 0.60  -0.75   3.58  -3.37  -0.22   0.00
-0.01   0.85  -0.20   0.45   0.00  -0.65
```

(4)

Twist screw, $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$.

```
  Entity
  (1,1)     (2,1)     (3,1)     (4,1)     (5,1)     (6,1)
-0.00025   0.05577   0.00041  -0.00146  -0.00001  -0.00000
-0.00025   0.05699   0.00042  -0.00149  -0.00001  -0.00000
-0.00025   0.05612   0.00041  -0.00151  -0.00001  -0.00000
-0.00025   0.05612   0.00041  -0.00151  -0.00001  -0.00000
-0.00025   0.05612   0.00041  -0.00151  -0.00001  -0.00000
-0.00025   0.05612   0.00041  -0.00151  -0.00001  -0.00000
-0.00025   0.05647   0.00041  -0.00155  -0.00001  -0.00000
  Entity
  (1,2)     (2,2)     (3,2)     (4,2)     (5,2)     (6,2)
-0.00809  -0.00480  -0.03906   0.00000   0.00000   0.00000
-0.00809  -0.00480  -0.03906   0.00000   0.00000   0.00000
-0.00799  -0.00509  -0.03857   0.00005   0.00000   0.00000
-0.00809  -0.00480  -0.03906   0.00000   0.00000   0.00000
-0.00786  -0.00477  -0.03889  -0.00000   0.00003  -0.00000
-0.00786  -0.00477  -0.03889  -0.00000   0.00003  -0.00000
```

-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03744	0.00151	0.00824	-0.00005	0.00087	-0.00011
0.03732	0.00155	0.00855	-0.00005	0.00084	-0.00010
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03823	0.00153	0.00842	-0.00005	0.00089	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.02198	-0.00250	-0.01688	0.00006	-0.00296	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02244	-0.00219	-0.01724	0.00001	-0.00302	0.00037

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
0.00009	-0.02201	0.00003	0.00297	0.00001	0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	0.00000
0.00009	-0.02096	0.00003	0.00283	0.00001	0.00000
0.00009	-0.02131	0.00003	0.00287	0.00001	0.00000
0.00009	-0.02131	0.00003	0.00287	0.00001	0.00000

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.00000	0.00035	0.00000	-0.00005	-0.00000	-0.00314
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00314
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00320
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327
-0.00000	0.00035	0.00000	-0.00005	-0.00000	-0.00314
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00314
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00320

Wrench screw, $\delta\underline{w}=[\delta\underline{f};\ \delta\underline{m}_0]$

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.01	1.63	0.07	-5.78	0.06	-0.15
-0.01	1.55	0.09	-5.53	0.02	-0.07
0.01	1.56	0.10	-5.58	0.09	-0.10
-0.01	1.54	0.10	-5.62	0.00	-0.06
0.00	1.54	0.10	-5.67	0.03	-0.08
-0.01	1.56	0.11	-5.66	0.03	-0.10
0.01	1.56	0.11	-5.69	0.04	-0.09

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.17	-0.18	-2.72	0.95	-0.38	0.05
-0.17	-0.17	-2.78	0.96	-0.36	0.05
-0.17	-0.19	-2.85	0.92	-0.34	0.06
-0.18	-0.18	-2.82	0.93	-0.34	0.06

-0.17	-0.20	-2.87	0.97	-0.37	0.06
-0.18	-0.21	-2.86	0.98	-0.36	0.05
-0.20	-0.20	-2.85	0.96	-0.35	0.05
-0.17	-0.19	-2.82	0.94	-0.33	0.05

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
1.03	0.16	0.70	0.38	2.92	-0.29
0.99	0.10	0.70	0.45	3.01	-0.24
1.00	0.08	0.72	0.49	3.01	-0.25
0.99	0.09	0.70	0.51	3.01	-0.27
1.01	0.09	0.72	0.52	3.03	-0.27
1.00	0.08	0.74	0.49	2.98	-0.23
1.00	0.10	0.72	0.43	2.97	-0.23
0.98	0.10	0.69	0.43	2.97	-0.22

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.87	-0.11	-1.25	0.10	-3.09	0.12
-0.82	-0.11	-1.20	0.09	-3.03	0.20
-0.83	-0.10	-1.20	0.09	-3.04	0.23
-0.83	-0.10	-1.24	0.09	-3.06	0.25
-0.82	-0.12	-1.23	0.10	-3.05	0.26
-0.84	-0.13	-1.28	0.10	-3.03	0.24
-0.83	-0.12	-1.23	0.10	-3.08	0.26
-0.84	-0.12	-1.26	0.10	-3.03	0.22

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.10	-0.80	-0.06	3.48	-0.08	0.01
-0.10	-0.75	-0.06	3.26	-0.09	0.02
-0.10	-0.87	-0.07	3.42	-0.09	0.01
-0.12	-0.81	-0.07	3.51	-0.10	0.02
-0.11	-0.78	-0.06	3.36	-0.10	0.04
-0.07	-0.81	-0.07	3.46	-0.09	0.05
-0.12	-0.66	-0.06	3.46	-0.08	0.04
-0.07	-0.99	-0.08	3.45	-0.10	0.02
-0.10	-0.74	-0.07	3.32	-0.09	0.05

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.01	-0.00	0.06	0.00	0.01	-0.69
-0.01	0.00	0.07	0.00	0.01	-0.68
-0.01	0.00	0.07	-0.00	0.01	-0.70
-0.00	0.01	0.06	-0.00	0.02	-0.69
-0.01	0.01	0.06	-0.00	0.02	-0.68
-0.01	0.01	0.06	0.00	0.01	-0.69
-0.01	0.01	0.06	0.00	0.01	-0.68
-0.01	0.01	0.06	-0.00	0.02	-0.70
-0.01	0.01	0.06	-0.00	0.01	-0.70

$$[\delta\underline{D}]_a=$$

$$\begin{matrix} -0.00025 & -0.00802 & 0.03753 & -0.02221 & 0.00009 & 0.00000 \\ 0.05624 & -0.00487 & 0.00130 & -0.00239 & -0.02162 & 0.00000 \\ 0.00041 & -0.03896 & 0.00830 & -0.01706 & 0.00003 & 0.00000 \\ -0.00151 & 0.00001 & -0.00002 & 0.00004 & 0.00292 & 0.00000 \\ -0.00001 & 0.00001 & 0.00087 & -0.00299 & 0.00001 & 0.00000 \\ 0.00000 & 0.00000 & -0.00011 & 0.00036 & 0.00000 & -0.00288 \end{matrix}$$

$$[\delta\underline{w}]_a= \begin{matrix} -0.00 & -0.18 & 1.00 & -0.83 & -0.10 & -0.01 \\ 1.56 & -0.19 & 0.10 & -0.11 & -0.80 & 0.01 \\ 0.10 & -2.82 & 0.71 & -1.24 & -0.07 & 0.06 \\ -5.65 & 0.95 & 0.46 & 0.10 & 3.41 & 0.00 \\ 0.04 & -0.35 & 2.99 & -3.05 & -0.09 & 0.01 \\ -0.09 & 0.05 & -0.25 & 0.22 & 0.03 & -0.62 \end{matrix}$$

$$[Ka]= \begin{matrix} 24.42 & -1.07 & -0.36 & -42.96 & 100.53 & 2.78 \\ 1.49 & 25.49 & 1.35 & -85.83 & -2.58 & -2.09 \\ 3.61 & 0.73 & 71.58 & -18.24 & -24.90 & -19.46 \\ 19.45 & -85.92 & -17.51 & 533.18 & -0.82 & 0.00 \\ 68.12 & 0.22 & -4.88 & -33.50 & 541.18 & -4.17 \\ -5.91 & -1.76 & 0.06 & -2.92 & -3.38 & 215.85 \end{matrix}$$

$$[\delta \underline{w}]_c= \begin{matrix} -0.02 & -0.19 & 0.95 & -0.77 & 0.03 & 0.00 \\ 1.44 & -0.11 & 0.04 & -0.09 & -0.75 & 0.00 \\ 0.03 & -2.76 & 0.59 & -1.22 & 0.00 & 0.06 \\ -5.48 & 0.33 & 0.29 & -0.02 & 3.25 & 0.00 \\ 0.59 & -0.73 & 3.60 & -3.36 & -0.22 & 0.00 \\ -0.01 & 0.85 & -0.20 & 0.45 & 0.00 & -0.58 \end{matrix}$$

(5) Twist screw, $\delta\underline{D}=[\delta\underline{x}; \delta\underline{\theta}]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.00026	0.05769	0.00042	-0.00159	-0.00001	-0.00000
-0.00026	0.05769	0.00042	-0.00159	-0.00001	-0.00000
-0.00025	0.05647	0.00041	-0.00155	-0.00001	-0.00000
-0.00025	0.05647	0.00041	-0.00155	-0.00001	-0.00000
-0.00025	0.05577	0.00041	-0.00146	-0.00001	-0.00000
-0.00025	0.05647	0.00041	-0.00155	-0.00001	-0.00000

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00833	-0.00482	-0.03924	0.00000	-0.00003	0.00000

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03768	0.00119	0.00842	-0.00000	0.00090	-0.00011
0.03644	0.00112	0.00789	-0.00000	0.00082	-0.00010
0.03722	0.00079	0.00807	0.00004	0.00084	-0.00010
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02211	-0.00211	-0.01657	0.00001	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02244	-0.00219	-0.01724	0.00001	-0.00302	0.00037
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02131	0.00003	0.00287	0.00001	0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02096	0.00003	0.00283	0.00001	0.00000
0.00009	-0.02131	0.00003	0.00287	0.00001	-0.00000

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
0.00000	0.00000	0.00000	0.00000	0.00000	-0.00314
0.00000	-0.00003	0.00000	0.00005	0.00000	-0.00314
0.00000	-0.00003	0.00000	0.00005	0.00000	-0.00314
0.00000	0.00000	0.00000	-0.00000	-0.00000	-0.00320
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
0.00000	0.00000	0.00000	-0.00000	-0.00000	-0.00320

Wrench screw, $\delta\underline{w}=[\delta\underline{f};\ \delta\underline{m}_0]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
0.00	1.56	0.09	-5.65	0.07	-0.08
-0.00	1.58	0.09	-5.61	0.05	-0.09
0.00	1.58	0.11	-5.67	0.08	-0.10
-0.01	1.57	0.10	-5.65	0.06	-0.09
0.00	1.61	0.11	-5.71	0.07	-0.10
0.00	1.60	0.12	-5.67	0.06	-0.10

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.17	-0.18	-2.78	0.94	-0.44	0.05
-0.19	-0.19	-2.80	0.94	-0.46	0.05
-0.19	-0.18	-2.80	0.94	-0.44	0.05
-0.21	-0.19	-2.83	0.93	-0.44	0.04
-0.22	-0.20	-2.85	0.91	-0.45	0.04
-0.19	-0.19	-2.82	0.91	-0.46	0.05
-0.21	-0.19	-2.83	0.91	-0.47	0.05

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.96	0.11	0.70	0.52	3.12	-0.26
0.99	0.11	0.70	0.53	3.25	-0.25
1.00	0.12	0.71	0.51	3.25	-0.27
0.99	0.12	0.71	0.54	3.25	-0.28
0.99	0.11	0.68	0.51	3.20	-0.25
0.96	0.11	0.66	0.51	3.04	-0.25
0.94	0.11	0.65	0.51	3.10	-0.25
0.99	0.11	0.68	0.51	3.17	-0.26

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.80	-0.17	-1.21	0.09	-3.17	0.24
-0.79	-0.18	-1.19	0.09	-3.18	0.22
-0.79	-0.18	-1.18	0.09	-3.15	0.24
-0.79	-0.18	-1.19	0.09	-3.16	0.22
-0.80	-0.18	-1.20	0.09	-3.15	0.26
-0.80	-0.18	-1.22	0.09	-3.22	0.23
-0.80	-0.18	-1.17	0.09	-3.17	0.24
-0.81	-0.18	-1.20	0.09	-3.22	0.26

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.08	-0.80	-0.05	3.45	-0.10	0.01
-0.10	-0.81	-0.09	3.49	-0.10	0.00
-0.10	-0.78	-0.05	3.49	-0.09	0.04
-0.08	-0.80	-0.09	3.44	-0.09	0.04
-0.11	-0.77	-0.07	3.39	-0.08	0.03
-0.08	-0.81	-0.09	3.30	-0.10	0.02

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.01	-0.01	0.06	0.03	-0.00	-0.80
-0.01	-0.01	0.05	0.03	0.00	-0.78
-0.01	-0.01	0.04	0.03	0.00	-0.81
-0.01	-0.01	0.05	0.03	-0.00	-0.78
-0.01	-0.01	0.05	0.03	0.00	-0.77

-0.01	-0.01	0.04	0.04	0.01	-0.80

$[\delta\underline{D}]_a=$

-0.00025	-0.00812	0.03732	-0.02223	0.00009	0.00000
0.05676	-0.00465	0.00089	-0.00238	-0.02160	-0.00001
0.00041	-0.03909	0.00820	-0.01702	0.00003	0.00000
-0.00155	-0.00002	0.00003	0.00004	0.00291	0.00002
-0.00001	-0.00000	0.00086	-0.00299	0.00001	0.00000
0.00000	0.00000	-0.00011	0.00036	0.00000	-0.00318

$[\delta\underline{w}]_a=$

-0.00	-0.20	0.98	-0.80	-0.09	-0.01
1.58	-0.19	0.11	-0.18	-0.80	-0.01
0.10	-2.82	0.69	-1.19	-0.07	0.05
-5.66	0.93	0.52	0.09	3.43	0.03
0.07	-0.45	3.17	-3.18	-0.09	0.00
-0.09	0.05	-0.26	0.24	0.02	-0.79

[Ka]=

24.23	-1.00	0.14	-39.89	86.34	2.94
1.76	25.63	1.45	-82.95	17.16	2.63
3.49	0.76	71.24	-20.27	-34.53	-15.30
19.31	-84.52	-17.94	549.06	2.62	-6.81
73.42	0.77	-3.84	-30.43	537.04	-0.69
-6.13	-1.82	0.29	-5.27	-4.56	248.28

$[\delta\underline{w}]_c=$

-0.02	-0.19	0.94	-0.77	0.03	0.00
1.46	-0.11	0.03	-0.09	-0.75	-0.00
0.03	-2.77	0.58	-1.21	0.00	0.07
-5.54	0.29	0.35	-0.02	3.25	0.01
0.59	-0.74	3.58	-3.36	-0.22	0.00
-0.01	0.85	-0.20	0.45	0.00	-0.64

(6)
Twist screw, $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.00026	0.05891	0.00043	-0.00162	-0.00001	-0.00000
-0.00025	0.05647	0.00041	-0.00155	-0.00001	-0.00000
-0.00026	0.05734	0.00042	-0.00154	-0.00001	-0.00000
-0.00025	0.05612	0.00041	-0.00151	-0.00001	-0.00000
-0.00026	0.05734	0.00042	-0.00154	-0.00001	-0.00000
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000
-0.00025	0.05612	0.00041	-0.00151	-0.00001	-0.00000
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00479	-0.03906	0.00000	-0.00000	0.00006
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00809	-0.00445	-0.03906	-0.00005	-0.00000	0.00000
-0.00833	-0.00482	-0.03924	0.00000	-0.00003	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03800	0.00116	0.00824	-0.00000	0.00086	-0.00011

0.03823	0.00119	0.00842	-0.00000	0.00089	-0.00011
0.03744	0.00151	0.00824	-0.00005	0.00087	-0.00011
0.03823	0.00119	0.00842	-0.00000	0.00089	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00017
0.03823	0.00119	0.00842	-0.00000	0.00089	-0.00011

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.02244	-0.00219	-0.01724	0.00001	-0.00302	0.00037
-0.02244	-0.00220	-0.01723	0.00001	-0.00302	0.00037
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02244	-0.00220	-0.01723	0.00001	-0.00302	0.00037

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02096	0.00003	0.00283	0.00001	-0.00006
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00008	-0.01974	0.00004	0.00280	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
0.00000	-0.00003	0.00000	0.00005	0.00000	-0.00320
0.00000	0.00000	0.00000	-0.00000	-0.00000	-0.00320
-0.00000	0.00003	0.00000	-0.00005	-0.00000	-0.00327
0.00000	-0.00003	0.00000	0.00005	0.00000	-0.00314
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327

Wrench screw, $\delta\underline{w}=[\delta\underline{f}; \delta\underline{m}_0]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
0.02	1.56	0.12	-5.84	0.08	-0.10
0.02	1.59	0.13	-6.00	0.12	-0.16
0.02	1.56	0.12	-5.84	0.07	-0.11
0.02	1.56	0.12	-5.90	0.06	-0.09
0.02	1.61	0.13	-5.86	0.14	-0.17
0.01	1.60	0.12	-5.90	0.12	-0.15
0.00	1.58	0.12	-5.88	0.07	-0.11
0.00	1.61	0.13	-5.96	0.08	-0.16

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.20	-0.19	-2.79	0.77	-0.49	0.04
-0.20	-0.19	-2.81	0.84	-0.49	0.04
-0.20	-0.20	-2.85	0.89	-0.45	0.04
-0.20	-0.19	-2.84	0.84	-0.41	0.04
-0.20	-0.19	-2.83	0.84	-0.41	0.05
-0.20	-0.20	-2.85	0.87	-0.41	0.05
-0.20	-0.20	-2.84	0.91	-0.44	0.04
-0.20	-0.20	-2.84	0.94	-0.46	0.04
-0.21	-0.21	-2.85	0.93	-0.47	0.04

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
1.00	0.11	0.70	0.39	3.13	-0.24
0.99	0.12	0.71	0.41	3.18	-0.26
0.98	0.12	0.74	0.43	3.09	-0.26
0.98	0.11	0.74	0.47	3.11	-0.29
0.99	0.10	0.72	0.49	3.03	-0.27

0.99	0.11	0.72	0.49	3.12	-0.28
0.98	0.11	0.70	0.45	3.07	-0.27

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.83	-0.15	-1.21	0.10	-3.31	0.20
-0.85	-0.15	-1.18	0.10	-3.17	0.25
-0.85	-0.15	-1.20	0.10	-3.18	0.26
-0.84	-0.15	-1.21	0.10	-3.20	0.24
-0.85	-0.15	-1.17	0.10	-3.21	0.27
-0.85	-0.16	-1.19	0.11	-3.19	0.24
-0.84	-0.15	-1.18	0.10	-3.19	0.26

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.06	-0.81	-0.07	3.41	-0.08	0.01
-0.04	-0.83	-0.06	3.51	-0.09	0.01
-0.04	-0.82	-0.07	3.56	-0.09	0.02
-0.06	-0.78	-0.05	3.47	-0.09	0.03
-0.07	-0.79	-0.05	3.42	-0.08	0.03
-0.07	-0.80	-0.07	3.51	-0.10	0.02
-0.07	-0.81	-0.07	3.54	-0.10	0.01

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.01	-0.01	0.09	0.03	-0.01	-0.78
-0.01	-0.01	0.07	0.03	-0.01	-0.76
-0.01	-0.01	0.08	0.03	-0.01	-0.75
-0.01	-0.01	0.08	0.03	-0.01	-0.77
-0.01	-0.01	0.07	0.03	-0.00	-0.79
-0.01	-0.01	0.07	0.03	-0.00	-0.75

$[\delta\underline{D}]_a=$

-0.00025	-0.00812	0.03786	-0.02231	0.00009	0.00000
0.05703	-0.00476	0.00122	-0.00218	-0.02144	-0.00001
0.00042	-0.03908	0.00832	-0.01713	0.00003	0.00000
-0.00153	-0.00001	-0.00001	0.00001	0.00291	0.00001
-0.00001	-0.00000	0.00088	-0.00300	0.00001	0.00000
0.00000	0.00001	-0.00012	0.00036	-0.00001	-0.00323

$[\delta\underline{w}]_a=$

0.01	-0.20	0.99	-0.84	-0.06	-0.01
1.58	-0.20	0.11	-0.15	-0.81	-0.01
0.12	-2.83	0.72	-1.19	-0.06	0.08
-5.90	0.87	0.45	0.10	3.49	0.03
0.09	-0.45	3.10	-3.21	-0.09	-0.01
-0.13	0.04	-0.27	0.25	0.02	-0.77

[Ka]=

23.60	-0.26	0.27	-23.08	104.77	3.04
1.49	25.34	1.65	-90.26	11.62	2.83
4.23	1.32	71.46	-12.71	-46.35	-23.81
18.56	-88.55	-15.40	545.89	-18.58	-7.75
69.46	1.39	-3.18	-24.70	569.34	2.00
-6.15	-2.67	0.57	-12.34	-8.62	237.70

$[\delta\underline{w}]_c=$

-0.02	-0.19	0.96	-0.77	0.03	0.00
1.46	-0.11	0.04	-0.08	-0.75	0.00
0.03	-2.77	0.59	-1.22	0.00	0.07
-5.56	0.31	0.31	-0.06	3.23	0.00
0.60	-0.74	3.63	-3.37	-0.22	0.00
-0.01	0.86	-0.21	0.45	0.00	-0.65

(7)
Twist screw, $\delta\underline{D}=[\delta\underline{x}; \delta\underline{\theta}]$.

```
  Entity
   (1,1)     (2,1)     (3,1)     (4,1)     (5,1)     (6,1)
 -0.00026  0.05734  0.00042  -0.00154  -0.00001  -0.00000
 -0.00025  0.05699  0.00042  -0.00149  -0.00001  -0.00000
 -0.00025  0.05699  0.00042  -0.00149  -0.00001  -0.00000
 -0.00026  0.05769  0.00042  -0.00159  -0.00001  -0.00000
 -0.00026  0.05769  0.00042  -0.00159  -0.00001  -0.00000
 -0.00025  0.05699  0.00042  -0.00149  -0.00001  -0.00000
 -0.00026  0.05734  0.00042  -0.00154  -0.00001  -0.00000
  Entity
   (1,2)     (2,2)     (3,2)     (4,2)     (5,2)     (6,2)
 -0.00833  -0.00482  -0.03924   0.00000  -0.00003   0.00000
 -0.00833  -0.00482  -0.03924   0.00000  -0.00003   0.00000
 -0.00786  -0.00477  -0.03889  -0.00000   0.00003  -0.00000
 -0.00833  -0.00447  -0.03924  -0.00005  -0.00003   0.00000
 -0.00833  -0.00482  -0.03924   0.00000  -0.00003   0.00000
 -0.00809  -0.00445  -0.03906  -0.00005  -0.00000   0.00000
 -0.00809  -0.00445  -0.03906  -0.00005  -0.00000   0.00000
 -0.00799  -0.00473  -0.03857   0.00000   0.00000   0.00000
  Entity
   (1,3)     (2,3)     (3,3)     (4,3)     (5,3)     (6,3)
  0.03823   0.00153   0.00842  -0.00005   0.00089  -0.00011
  0.03744   0.00151   0.00824  -0.00005   0.00087  -0.00011
  0.03721   0.00149   0.00807  -0.00005   0.00084  -0.00010
  0.03768   0.00119   0.00842  -0.00000   0.00090  -0.00011
  0.03768   0.00119   0.00842  -0.00000   0.00090  -0.00011
  0.03745   0.00116   0.00824  -0.00000   0.00087  -0.00011
  0.03745   0.00116   0.00824  -0.00000   0.00087  -0.00011
  Entity
   (1,4)     (2,4)     (3,4)     (4,4)     (5,4)     (6,4)
 -0.02244  -0.00254  -0.01723   0.00006  -0.00302   0.00037
 -0.02221  -0.00217  -0.01706   0.00001  -0.00299   0.00036
 -0.02244  -0.00219  -0.01724   0.00001  -0.00302   0.00037
 -0.02221  -0.00217  -0.01706   0.00001  -0.00299   0.00036
 -0.02244  -0.00220  -0.01723   0.00001  -0.00302   0.00037
 -0.02198  -0.00180  -0.01688  -0.00003  -0.00296   0.00036
 -0.02299  -0.00220  -0.01723   0.00001  -0.00301   0.00037
  Entity
   (1,5)     (2,5)     (3,5)     (4,5)     (5,5)     (6,5)
  0.00032  -0.02199   0.00021   0.00297   0.00004  -0.00000
  0.00032  -0.02164   0.00021   0.00292   0.00004  -0.00000
  0.00032  -0.02164   0.00021   0.00292   0.00004  -0.00000
  0.00009  -0.02166   0.00003   0.00292   0.00001  -0.00000
  0.00009  -0.02201   0.00003   0.00297   0.00001  -0.00000
  0.00009  -0.02131   0.00003   0.00287   0.00001  -0.00000
  0.00009  -0.02131   0.00003   0.00287   0.00001  -0.00000
  Entity
   (1,6)     (2,6)     (3,6)     (4,6)     (5,6)     (6,6)
  0.00000   0.00000   0.00000   0.00000  -0.00000  -0.00314
  0.00000   0.00000   0.00000  -0.00000   0.00000  -0.00326
  0.00000  -0.00035   0.00000   0.00005   0.00000  -0.00314
  0.00000   0.00000   0.00000   0.00000  -0.00000  -0.00327
 -0.00000   0.00035   0.00000  -0.00005  -0.00000  -0.00320
 -0.00000   0.00035   0.00000  -0.00005  -0.00000  -0.00320
  0.00000   0.00000   0.00000  -0.00000   0.00000  -0.00326
```

Wrench screw, $\delta\underline{w}=[\delta\underline{f}; \delta\underline{m}_0]$.

```
  Entity
   (1,1)     (2,1)     (3,1)     (4,1)     (5,1)     (6,1)
```

```
 -0.01    1.58    0.03   -5.71    0.07   -0.19
 -0.01    1.60    0.02   -5.74    0.09   -0.19
 -0.01    1.59    0.02   -5.71    0.07   -0.17
 -0.01    1.59    0.02   -5.74    0.11   -0.16
 -0.03    1.60    0.03   -5.78    0.07   -0.18
 -0.01    1.59    0.02   -5.75    0.06   -0.19
 -0.01    1.59    0.03   -5.70    0.09   -0.18
  Entity
  (1,2)   (2,2)   (3,2)   (4,2)   (5,2)   (6,2)
 -0.17   -0.19   -2.81    0.86   -0.43    0.06
 -0.17   -0.19   -2.81    0.88   -0.41    0.06
 -0.17   -0.19   -2.83    0.92   -0.39    0.06
 -0.17   -0.21   -2.82    0.94   -0.42    0.05
 -0.17   -0.21   -2.83    0.95   -0.38    0.05
 -0.17   -0.19   -2.81    0.93   -0.38    0.05
 -0.17   -0.20   -2.81    0.92   -0.38    0.05
 -0.17   -0.19   -2.81    0.91   -0.40    0.04
  Entity
  (1,3)   (2,3)   (3,3)   (4,3)   (5,3)   (6,3)
  0.94    0.10    0.69    0.47    2.88   -0.29
  0.93    0.11    0.62    0.46    2.79   -0.32
  0.96    0.11    0.70    0.48    2.89   -0.31
  0.94    0.11    0.64    0.46    2.81   -0.30
  0.95    0.12    0.69    0.47    2.82   -0.31
  0.96    0.11    0.69    0.47    2.88   -0.28
  0.96    0.11    0.70    0.47    2.88   -0.29
  Entity
  (1,4)   (2,4)   (3,4)   (4,4)   (5,4)   (6,4)
-0.83   -0.15   -1.21    0.14   -3.09    0.19
 -0.86   -0.16   -1.23    0.15   -3.01    0.17
 -0.84   -0.16   -1.20    0.15   -2.97    0.19
 -0.84   -0.16   -1.20    0.16   -2.96    0.21
 -0.84   -0.16   -1.19    0.16   -2.98    0.21
 -0.83   -0.16   -1.20    0.16   -2.95    0.22
 -0.84   -0.15   -1.19    0.16   -2.97    0.24
  Entity
  (1,5)   (2,5)   (3,5)   (4,5)   (5,5)   (6,5)
 -0.04   -0.81   -0.06    3.50   -0.09    0.03
 -0.06   -0.76   -0.04    3.31   -0.08    0.03
 -0.03   -0.83   -0.06    3.60   -0.10    0.03
 -0.07   -0.75   -0.07    3.24   -0.09    0.06
 -0.08   -0.72   -0.05    3.53   -0.08    0.03
 -0.06   -0.79   -0.07    3.42   -0.10    0.05
 -0.06   -0.69   -0.05    3.41   -0.09    0.04
  Entity
  (1,6)   (2,6)   (3,6)   (4,6)   (5,6)   (6,6)
 -0.01   -0.01    0.08    0.01   -0.02   -0.74
 -0.00   -0.02    0.10    0.03    0.00   -0.75
 -0.00   -0.00    0.09    0.02   -0.00   -0.78
 -0.01   -0.00    0.09    0.02   -0.00   -0.78
  0.00   -0.00    0.09    0.03    0.01   -0.73
 -0.01   -0.00    0.11    0.03   -0.00   -0.78
 -0.00   -0.00    0.08    0.02    0.01   -0.78

[Dm]=
 -0.00026  -0.00817   0.03759  -0.02239   0.00019   0.00000
  0.05729  -0.00467   0.00132  -0.00218  -0.02165   0.00006
  0.00042  -0.03907   0.00829  -0.01713   0.00011   0.00000
 -0.00153  -0.00002  -0.00002   0.00001   0.00292  -0.00001
 -0.00001  -0.00001   0.00088  -0.00300   0.00002   0.00000
  0.00000   0.00000  -0.00011   0.00037   0.00000  -0.00375
```

[wm]=

-0.01	-0.17	0.95	-0.84	-0.06	-0.01
1.59	-0.20	0.11	-0.16	-0.76	-0.01
0.02	-2.82	0.68	-1.20	-0.06	0.11
-5.73	0.91	0.47	0.15	3.43	0.03
0.08	-0.40	2.85	-2.99	-0.09	0.00
-0.18	0.05	-0.30	0.20	0.04	-0.89

[Ka]=

22.68	-0.86	-0.31	-28.27	113.14	1.38
1.28	25.90	1.69	-69.93	14.32	1.89
2.86	-0.86	71.61	-28.47	-32.33	-28.43
20.61	-85.37	-17.75	541.29	-40.88	-9.65
63.97	0.94	-3.42	-32.04	537.78	0.09
-7.73	-3.51	0.69	-12.50	17.10	237.62

[wc]=

-0.02	-0.19	0.95	-0.78	0.04	0.00
1.47	-0.11	0.04	-0.08	-0.75	0.00
0.03	-2.77	0.59	-1.22	0.01	0.08
-5.58	0.29	0.29	-0.06	3.26	0.01
0.60	-0.74	3.61	-3.38	-0.21	0.00
-0.01	0.85	-0.20	0.45	0.00	-0.76

(8)Twist screw, $\delta\underline{D}=[\delta\underline{x};\ \delta\underline{\theta}]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.00026	0.05891	0.00043	-0.00162	-0.00001	-0.00000
-0.00026	0.05856	0.00043	-0.00157	-0.00001	-0.00000
-0.00026	0.05734	0.00042	-0.00154	-0.00001	-0.00000
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000
-0.00026	0.05856	0.00043	-0.00157	-0.00001	-0.00000
-0.00026	0.05891	0.00043	-0.00162	-0.00001	-0.00000

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00479	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00515	-0.03906	0.00005	0.00000	0.00000
-0.00809	-0.00479	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00479	-0.03906	0.00000	0.00000	0.00000

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03768	0.00119	0.00842	-0.00000	0.00090	-0.00011
0.03744	0.00151	0.00824	-0.00005	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03744	0.00151	0.00824	-0.00005	0.00087	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.02198	-0.00215	-0.01688	0.00001	-0.00296	0.00036
-0.02221	-0.00182	-0.01706	-0.00003	-0.00299	0.00036
-0.02244	-0.00185	-0.01723	-0.00003	-0.00302	0.00037
-0.02244	-0.00220	-0.01723	0.00001	-0.00302	0.00037
-0.02244	-0.00219	-0.01724	0.00001	-0.00302	0.00037
-0.02244	-0.00219	-0.01724	0.00001	-0.00302	0.00037
-0.02244	-0.00254	-0.01723	0.00006	-0.00302	0.00037

-0.02198	-0.00250	-0.01688	0.00006	-0.00296	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00006
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02131	0.00003	0.00287	0.00001	-0.00000
0.00009	-0.02096	0.00003	0.00283	0.00001	0.00000

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.00000	0.00035	0.00000	-0.00005	-0.00000	-0.00314
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00327
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00327
-0.00000	0.00035	0.00000	-0.00005	-0.00000	-0.00314
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00320
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	0.00000	-0.00314
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00327
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00314

Wrench screw, $\delta\underline{w}=[\delta\underline{f}; \delta\underline{m}_0]$

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.01	1.59	0.11	-5.79	0.06	-0.18
-0.01	1.59	0.12	-5.88	0.09	-0.17
-0.01	1.56	0.12	-5.85	0.07	-0.15
0.00	1.54	0.12	-5.76	0.09	-0.14
0.00	1.59	0.11	-5.83	0.12	-0.17
0.00	1.59	0.12	-5.84	0.10	-0.19

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.19	-0.20	-2.85	0.86	-0.54	0.07
-0.17	-0.20	-2.81	0.89	-0.51	0.07
-0.18	-0.21	-2.85	0.93	-0.49	0.07
-0.18	-0.19	-2.85	0.89	-0.50	0.07
-0.18	-0.20	-2.82	0.87	-0.50	0.07
-0.19	-0.20	-2.85	0.88	-0.53	0.07
-0.20	-0.21	-2.87	0.86	-0.53	0.06

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.98	0.11	0.63	0.43	3.22	-0.26
0.98	0.11	0.64	0.43	2.91	-0.26
0.96	0.10	0.62	0.42	3.06	-0.25
0.96	0.11	0.61	0.45	3.01	-0.27
0.96	0.11	0.61	0.45	2.85	-0.28
0.98	0.11	0.65	0.46	3.04	-0.26
0.98	0.09	0.65	0.47	3.08	-0.27
0.99	0.08	0.63	0.45	3.00	-0.24

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.85	-0.15	-1.22	0.15	-3.18	0.12
-0.85	-0.15	-1.22	0.15	-3.13	0.19
-0.85	-0.16	-1.20	0.17	-3.20	0.18
-0.84	-0.15	-1.21	0.16	-3.12	0.17
-0.84	-0.16	-1.22	0.17	-3.12	0.17
-0.85	-0.16	-1.23	0.15	-3.12	0.14
-0.87	-0.16	-1.22	0.19	-3.18	0.17
-0.85	-0.16	-1.19	0.17	-3.14	0.15
-0.86	-0.16	-1.23	0.18	-3.18	0.17

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.02	-0.76	-0.03	3.34	-0.03	-0.01
-0.01	-0.78	-0.04	3.40	-0.07	0.03
-0.01	-0.75	-0.04	3.52	-0.07	0.03
-0.01	-0.76	-0.04	3.40	-0.08	0.03
-0.01	-0.80	-0.03	3.50	-0.09	0.02
-0.01	-0.80	-0.04	3.52	-0.08	0.03
-0.01	-0.00	0.05	0.01	-0.02	-0.75

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
-0.00	-0.01	0.08	0.03	0.00	-0.74
-0.00	-0.00	0.09	0.02	-0.00	-0.79
-0.00	-0.00	0.10	0.02	0.00	-0.79
-0.01	-0.00	0.09	0.02	-0.01	-0.79
-0.00	-0.00	0.07	0.02	0.00	-0.77
-0.00	-0.00	0.09	0.02	-0.00	-0.79
0.00	-0.00	0.09	0.02	0.01	-0.76
0.00	0.00	0.08	0.02	0.01	-0.77

$$[Dm]= \begin{matrix} -0.00026 & -0.00158 & 0.03748 & -0.02229 & 0.00009 & 0.00000 \\ 0.05821 & -0.00409 & 0.00112 & -0.00218 & -0.02154 & -0.00008 \\ 0.00043 & -0.03230 & 0.00826 & -0.01712 & 0.00003 & 0.00000 \\ -0.00157 & 0.00002 & 0.00000 & 0.00001 & 0.00290 & 0.00001 \\ -0.00001 & 0.00012 & 0.00087 & -0.00300 & 0.00001 & 0.00000 \\ 0.00000 & -0.00002 & -0.00011 & 0.00037 & -0.00001 & -0.00320 \end{matrix}$$

$$[wm]= \begin{matrix} -0.01 & -0.02 & 0.97 & -0.85 & -0.01 & -0.00 \\ 1.58 & -0.16 & 0.10 & -0.16 & -0.77 & -0.00 \\ 0.12 & -2.35 & 0.63 & -1.22 & -0.04 & 0.08 \\ -5.83 & 0.82 & 0.45 & 0.17 & 3.45 & 0.02 \\ 0.09 & 0.02 & 3.02 & -3.15 & -0.07 & -0.00 \\ -0.17 & 0.02 & -0.26 & 0.16 & 0.02 & -0.77 \end{matrix}$$

$$[Ka]= \begin{matrix} 23.45 & -0.12 & -0.18 & -6.04 & 110.69 & 0.68 \\ 1.29 & 24.86 & 1.66 & -82.55 & 14.75 & -0.54 \\ 1.23 & 1.39 & 72.31 & -3.10 & -20.71 & -25.70 \\ 18.77 & -84.88 & -15.41 & 556.73 & -43.04 & -2.25 \\ 68.08 & 1.47 & -2.10 & -17.22 & 555.89 & 0.25 \\ -6.83 & -3.33 & 0.07 & -16.28 & 27.97 & 241.01 \end{matrix}$$

$$[wc]= \begin{matrix} -0.02 & -0.03 & 0.95 & -0.77 & 0.03 & 0.00 \\ 1.49 & -0.10 & 0.04 & -0.08 & -0.75 & -0.00 \\ 0.03 & -2.29 & 0.59 & -1.22 & 0.00 & 0.07 \\ -5.68 & 0.34 & 0.32 & -0.05 & 3.24 & 0.01 \\ 0.61 & -0.12 & 3.60 & -3.37 & -0.22 & -0.00 \\ -0.01 & 0.70 & -0.20 & 0.45 & -0.00 & -0.65 \end{matrix}$$

(9)Twist screw, $\delta\underline{D}=[\delta\underline{x}; \delta\underline{\theta}]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.00026	0.05891	0.00043	-0.00162	-0.00001	-0.00000
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000
-0.00026	0.05821	0.00043	-0.00152	-0.00001	-0.00000
-0.00026	0.05856	0.00043	-0.00157	-0.00001	0.00006
-0.00025	0.05699	0.00042	-0.00149	-0.00001	-0.00000
-0.00026	0.05821	0.00043	-0.00152	-0.00001	-0.00000
-0.00026	0.05821	0.00043	-0.00152	-0.00001	-0.00000
-0.00026	0.05891	0.00043	-0.00162	-0.00001	-0.00000
-0.00026	0.05856	0.00043	-0.00157	-0.00001	-0.00000

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.00786	-0.00477	-0.03889	-0.00000	0.00003	-0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00786	-0.00477	-0.03889	-0.00000	0.00003	-0.00000
-0.00809	-0.00480	-0.03906	0.00000	0.00000	0.00000
-0.00809	-0.00479	-0.03906	0.00000	0.00000	0.00000

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03745	0.00116	0.00824	-0.00000	0.00087	-0.00011
0.03722	0.00079	0.00807	0.00004	0.00084	-0.00010
0.03745	0.00081	0.00824	0.00004	0.00087	-0.00011
0.03744	0.00151	0.00824	-0.00005	0.00087	-0.00017

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02198	-0.00250	-0.01688	0.00006	-0.00296	0.00036
-0.02221	-0.00252	-0.01706	0.00006	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02221	-0.00217	-0.01706	0.00001	-0.00299	0.00036
-0.02244	-0.00220	-0.01723	0.00001	-0.00302	0.00037
-0.02244	-0.00254	-0.01723	0.00006	-0.00302	0.00037

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
0.00009	-0.02131	0.00003	0.00287	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02131	0.00003	0.00287	0.00001	0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000
0.00009	-0.02201	0.00003	0.00297	0.00001	-0.00000

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00327
0.00000	0.00000	0.00000	-0.00000	-0.00000	-0.00320
0.00000	0.00000	0.00000	-0.00000	-0.00000	-0.00320
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327
0.00000	-0.00035	0.00000	0.00005	0.00000	-0.00314
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327

Wrench screw, $\delta\underline{w}=[\delta\underline{f}; \delta\underline{m}_0]$

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
0.00	1.56	0.10	-5.74	0.08	-0.17
0.00	1.65	0.10	-5.85	0.09	-0.16
0.00	1.58	0.10	-5.77	0.09	-0.16
-0.01	1.60	0.11	-5.84	0.08	-0.19
0.01	1.60	0.12	-5.79	0.14	-0.21
0.00	1.60	0.12	-5.85	0.08	-0.21
0.00	1.59	0.12	-5.80	0.10	-0.21
-0.01	1.59	0.11	-5.75	0.09	-0.20
0.00	1.58	0.12	-5.77	0.09	-0.20

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)

```
 -0.21   -0.20   -2.75    0.80   -0.55    0.06
 -0.21   -0.20   -2.77    0.83   -0.55    0.07
 -0.19   -0.21   -2.77    0.84   -0.52    0.08
 -0.19   -0.20   -2.73    0.87   -0.52    0.07
 -0.19   -0.19   -2.74    0.86   -0.51    0.07
 -0.19   -0.19   -2.73    0.83   -0.49    0.07
  Entity
  (1,3)   (2,3)   (3,3)   (4,3)   (5,3)   (6,3)
  0.96    0.08    0.71    0.46    3.06   -0.24
  0.99    0.11    0.71    0.45    3.19   -0.25
  0.96    0.10    0.69    0.44    2.98   -0.24
  0.98    0.11    0.72    0.46    3.07   -0.27
  0.98    0.10    0.72    0.46    3.04   -0.25
  0.98    0.10    0.73    0.48    3.09   -0.27
  0.94    0.09    0.70    0.47    2.90   -0.26
  0.98    0.07    0.71    0.50    3.10   -0.26
  Entity
  (1,4)   (2,4)   (3,4)   (4,4)   (5,4)   (6,4)
 -0.86   -0.18   -1.28    0.22   -3.23    0.17
 -0.84   -0.19   -1.23    0.24   -3.19    0.18
 -0.83   -0.19   -1.22    0.25   -3.17    0.22
 -0.84   -0.19   -1.24    0.24   -3.17    0.20
 -0.84   -0.19   -1.25    0.25   -3.18    0.23
 -0.85   -0.17   -1.24    0.24   -3.18    0.21
 -0.85   -0.18   -1.23    0.25   -3.19    0.23
 -0.86   -0.20   -1.26    0.26   -3.20    0.25
  Entity
  (1,5)   (2,5)   (3,5)   (4,5)   (5,5)   (6,5)
 -0.05   -0.73   -0.04    3.36   -0.09    0.03
 -0.03   -0.77   -0.04    3.36   -0.09    0.04
 -0.04   -0.75   -0.04    3.36   -0.08    0.04
 -0.06   -0.80   -0.04    3.60   -0.11    0.02
 -0.04   -0.80   -0.04    3.54   -0.10    0.03
 -0.07   -0.80   -0.04    3.58   -0.11    0.04
 -0.06   -0.77   -0.05    3.43   -0.09    0.05
 -0.05   -0.75   -0.04    3.39   -0.08    0.05
  Entity
  (1,6)   (2,6)   (3,6)   (4,6)   (5,6)   (6,6)
 -0.00   -0.02    0.07    0.04   -0.00   -0.76
  0.00   -0.00    0.07    0.03    0.01   -0.77
 -0.00   -0.01    0.09    0.03    0.00   -0.79
 -0.01   -0.01    0.09    0.03   -0.01   -0.79
  0.00   -0.00    0.08    0.03    0.01   -0.75
 -0.01   -0.00    0.09    0.03   -0.00   -0.77

[Dm]=
  -0.00026  -0.00801   0.03742  -0.02224   0.00009   0.00000
   0.05817  -0.00479   0.00098  -0.00235  -0.02166  -0.00006
   0.00043  -0.03900   0.00822  -0.01708   0.00003   0.00000
  -0.00155   0.00000   0.00002   0.00004   0.00292   0.00001
  -0.00001   0.00001   0.00087  -0.00299   0.00001   0.00000
   0.00001   0.00000  -0.00012   0.00036   0.00000  -0.00323
[wm]=
   -0.00  -0.20   0.97  -0.85  -0.05   0.00
    1.59  -0.20   0.10  -0.19  -0.77  -0.01
    0.11  -2.75   0.71  -1.24  -0.04   0.08
   -5.80   0.84   0.46   0.24   3.45   0.03
    0.09  -0.52   3.05  -3.19  -0.09   0.00
   -0.19   0.07  -0.26   0.21   0.04  -0.77
[Ka]=
          23.44   -0.48    0.31  -21.76  107.04     0.99
```

```
            0.96   25.38    1.78  -75.96   24.30    1.41
            3.98    1.27   69.49   -5.48  -14.65  -25.36
           19.18  -84.72  -15.05  553.71  -65.95   -6.86
           68.87    1.31   -0.75  -26.42  556.42   -0.61
           -6.11   -3.70   -0.08  -14.46    7.02  239.31
[wc]=
   -0.02   -0.19    0.95   -0.77    0.03    0.00
    1.49   -0.11    0.03   -0.09   -0.75   -0.00
    0.03   -2.76    0.58   -1.22    0.00    0.07
   -5.66    0.32    0.34   -0.03    3.26    0.01
    0.61   -0.72    3.59   -3.36   -0.22    0.00
   -0.01    0.85   -0.20    0.45    0.00   -0.65

(10)
Twist screw,  δD=[δx; δθ].
   Entity
    (1,1)     (2,1)     (3,1)     (4,1)     (5,1)     (6,1)
  -0.00026  0.05734   0.00042  -0.00154  -0.00001  -0.00000
  -0.00026  0.05821   0.00043  -0.00152  -0.00001  -0.00000
  -0.00026  0.05821   0.00043  -0.00152  -0.00001  -0.00000
  -0.00026  0.05891   0.00043  -0.00162  -0.00001  -0.00000
  -0.00026  0.05821   0.00043  -0.00152  -0.00001  -0.00000
  -0.00026  0.05943   0.00044  -0.00156  -0.00001  -0.00000
  -0.00026  0.05821   0.00043  -0.00152  -0.00001  -0.00000
   Entity
    (1,2)     (2,2)     (3,2)     (4,2)     (5,2)     (6,2)
  -0.00809  -0.00445  -0.03906  -0.00005  -0.00000   0.00000
  -0.00809  -0.00479  -0.03906   0.00000   0.00000   0.00000
  -0.00809  -0.00515  -0.03906   0.00005   0.00000   0.00000
  -0.00809  -0.00480  -0.03906   0.00000   0.00000   0.00000
  -0.00809  -0.00515  -0.03906   0.00005   0.00000   0.00000
  -0.00832  -0.00517  -0.03924   0.00005  -0.00003   0.00000
  -0.00809  -0.00480  -0.03906   0.00000   0.00000   0.00000
  -0.00809  -0.00480  -0.03906   0.00000   0.00000   0.00000
   Entity
    (1,3)     (2,3)     (3,3)     (4,3)     (5,3)     (6,3)
   0.03768   0.00153   0.00842  -0.00005   0.00090  -0.00011
   0.03745   0.00116   0.00824  -0.00000   0.00087  -0.00011
   0.03744   0.00151   0.00824  -0.00005   0.00087  -0.00011
   0.03800   0.00116   0.00824  -0.00000   0.00086  -0.00011
   0.03745   0.00116   0.00824  -0.00000   0.00087  -0.00011
   0.03823   0.00153   0.00842  -0.00005   0.00089  -0.00011
   0.03745   0.00116   0.00824  -0.00000   0.00087  -0.00011
   Entity
    (1,4)     (2,4)     (3,4)     (4,4)     (5,4)     (6,4)
  -0.02244  -0.00254  -0.01723   0.00006  -0.00302   0.00037
  -0.02244  -0.00220  -0.01723   0.00001  -0.00302   0.00037
  -0.02244  -0.00220  -0.01723   0.00001  -0.00302   0.00037
  -0.02221  -0.00217  -0.01706   0.00001  -0.00299   0.00036
  -0.02221  -0.00252  -0.01706   0.00006  -0.00299   0.00036
  -0.02221  -0.00252  -0.01706   0.00006  -0.00299   0.00036
  -0.02244  -0.00219  -0.01724   0.00001  -0.00302   0.00037
  -0.02221  -0.00182  -0.01706  -0.00003  -0.00299   0.00036
  -0.02221  -0.00217  -0.01706   0.00001  -0.00299   0.00036
   Entity
    (1,5)     (2,5)     (3,5)     (4,5)     (5,5)     (6,5)
   0.00009  -0.02166   0.00003   0.00292   0.00001  -0.00000
   0.00009  -0.02166   0.00003   0.00292   0.00001  -0.00000
   0.00009  -0.02131   0.00003   0.00287   0.00001  -0.00000
   0.00009  -0.02131   0.00003   0.00287   0.00001  -0.00000
   0.00009  -0.02166   0.00003   0.00292   0.00001   0.00006
```

0.00009	-0.02131	0.00003	0.00287	0.00001	-0.00000
0.00009	-0.02166	0.00003	0.00292	0.00001	-0.00000

Entity

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327
0.00000	0.00000	0.00000	-0.00000	-0.00000	-0.00320
0.00000	0.00000	0.00000	-0.00000	0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00333
-0.00000	0.00000	0.00000	-0.00005	-0.00000	-0.00320
0.00000	-0.00000	0.00000	0.00005	0.00000	-0.00327
0.00000	0.00000	0.00000	0.00000	-0.00000	-0.00314

Wrench screw, $\delta \underline{w}=[\delta \underline{f}; \delta \underline{m}_0]$.

Entity

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
-0.04	1.54	0.09	-5.71	0.24	-0.14
-0.05	1.53	0.07	-5.79	0.12	-0.16
-0.06	1.52	0.07	-5.76	0.09	-0.18
-0.06	1.52	0.06	-5.81	0.09	-0.15
-0.06	1.52	0.07	-5.85	0.12	-0.15
-0.06	1.50	0.07	-5.83	0.09	-0.14
-0.06	1.51	0.07	-5.80	0.15	-0.18

Entity

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
-0.19	-0.19	-2.81	0.77	-0.47	0.07
-0.17	-0.18	-2.87	0.78	-0.44	0.08
-0.19	-0.19	-2.90	0.80	-0.45	0.08
-0.17	-0.20	-2.90	0.80	-0.44	0.08
-0.18	-0.18	-2.89	0.80	-0.45	0.08
-0.17	-0.18	-2.90	0.76	-0.47	0.08
-0.18	-0.18	-2.90	0.80	-0.47	0.08
-0.19	-0.19	-2.97	0.80	-0.49	0.07

Entity

(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
1.02	0.15	0.75	0.46	3.08	-0.23
0.98	0.08	0.72	0.49	2.97	-0.21
0.97	0.09	0.72	0.47	2.93	-0.22
1.00	0.08	0.77	0.50	3.03	-0.24
1.00	0.08	0.77	0.52	3.05	-0.24
1.00	0.07	0.76	0.52	3.05	-0.25
1.00	0.09	0.77	0.52	3.14	-0.28

Entity

(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
-0.83	-0.14	-1.30	0.11	-3.37	0.06
-0.83	-0.14	-1.30	0.11	-3.06	0.09
-0.85	-0.14	-1.31	0.12	-3.06	0.09
-0.85	-0.14	-1.27	0.11	-3.10	0.10
-0.85	-0.14	-1.27	0.13	-3.07	0.12
-0.86	-0.14	-1.27	0.11	-3.12	0.12
-0.87	-0.14	-1.27	0.11	-3.14	0.09
-0.86	-0.14	-1.27	0.11	-3.21	0.04
-0.85	-0.13	-1.27	0.12	-3.10	0.12

Entity

(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
-0.01	-0.72	0.01	3.28	-0.02	0.00
-0.02	-0.73	-0.03	3.23	-0.06	0.01
-0.02	-0.75	-0.03	3.31	-0.07	0.01
-0.03	-0.73	-0.03	3.26	-0.08	0.01
-0.02	-0.72	-0.06	3.47	-0.08	0.04
-0.02	-0.77	-0.04	3.34	-0.08	0.02
-0.02	-0.72	-0.05	3.43	-0.08	0.04

```
   Entity
   (1,6)    (2,6)    (3,6)    (4,6)    (5,6)    (6,6)
  -0.01    -0.00     0.08     0.02    -0.01    -0.81
  -0.01    -0.00     0.08     0.02    -0.01    -0.80
  -0.01     0.00     0.07     0.01    -0.00    -0.78
  -0.01    -0.00     0.08     0.01    -0.01    -0.82
  -0.01     0.00     0.08     0.01    -0.01    -0.82
  -0.01     0.00     0.08     0.01    -0.01    -0.82
  -0.00     0.00     0.08     0.01     0.01    -0.78
[Dm]=
  -0.00026  -0.00812   0.03767  -0.02231   0.00009   0.00000
   0.05836  -0.00489   0.00132  -0.00226  -0.02151   0.00000
   0.00043  -0.03908   0.00829  -0.01714   0.00003   0.00000
  -0.00154   0.00001  -0.00002   0.00002   0.00290   0.00000
  -0.00001  -0.00000   0.00088  -0.00300   0.00001   0.00000
   0.00000   0.00000  -0.00011   0.00036   0.00001  -0.00324
[wm]=
  -0.06  -0.18   1.00  -0.85  -0.02  -0.01
   1.52  -0.19   0.09  -0.14  -0.73   0.00
   0.07  -2.89   0.75  -1.28  -0.03   0.08
  -5.79   0.79   0.50   0.11   3.33   0.01
   0.13  -0.46   3.04  -3.14  -0.07  -0.01
  -0.16   0.08  -0.24   0.09   0.02  -0.80
[Ka]=
   24.04   -1.29   -0.24  -17.62  106.96    2.65
    0.93   24.06    1.54  -74.86   11.92    0.00
    4.39    0.48   73.04   -8.54  -26.14  -24.25
   20.51  -85.47  -13.58  514.75  -45.35   -3.97
   68.30    2.37   -2.77   -9.58  551.16    1.76
   -6.68   -3.19   -0.21  -17.94   52.50  248.24
[wc]=
  -0.02  -0.19   0.95  -0.77   0.03   0.00
   1.49  -0.12   0.04  -0.09  -0.75   0.00
   0.03  -2.77   0.59  -1.22   0.00   0.07
  -5.68   0.33   0.29  -0.04   3.23   0.00
   0.61  -0.74   3.62  -3.37  -0.22   0.00
  -0.01   0.85  -0.20   0.45   0.00  -0.66

*****************************End*****************************
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APPENDIX B
A COMPARISON OF STIFFNESS MAPPINGS USING POTENTIAL ENERGY

1. Introduction

In the usual representation of a stiffness mapping, the stiffness matrix has four 3x3 sub-matrices and each sub-matrix contains elements of a different dimension. This lack of homogeneity is a disadvantage, because using matrix norms to compare stiffness mappings is, in such circumstances, void of physical meaning.

A concise formulation to express the stiffness mapping for a 6-DOF in-parallel compliant mechanism is derived in this section, resulting in elements of the stiffness matrix having the same dimension throughout the entire matrix. The potential energy of the system can be computed from this formulation and it can be used to compare quantitatively different stiffness mappings.

A stiffness mapping for a given 6-DOF in-parallel compliant mechanism is usually represented by a map that transforms a general twist into a general wrench in e(3), where e(3) is the Lie algebra of the Euclidean group. Both the twist and related wrench are represented as general screws in a basis of e(3), for example $\{\underline{i}, \underline{j}, \underline{k}; \underline{i}, \varepsilon\underline{j}, \varepsilon\underline{k}\}$, $(\varepsilon^2=0)$.

The stiffness matrix may be derived from the geometric configuration of the in-parallel mechanism [2]. Then, in the resulting stiffness matrix, the elements of the upper-left 3x3 submatrix have dimension $(F*L^{-1})^{+}$. The elements of the lower-left 3x3 submatrix have dimension F. The elements of the upper-right 3x3 submatrix have

---

[+] F represents the dimension of force, and L represents the dimension of length.

dimension F, and the elements of the lower-right 3x3 submatrix have dimension F*L. The quadratic form of a stiffness matrix, i.e. $\delta\underline{D}^T[K]\delta\underline{D}$, is the second derivative of potential energy for the in-parallel compliant mechanism [15, 24]. Hence, the computation of potential energy in the system can be obtained by integrating the quadratic form of [K] twice. The only invariant property of [K] is its signature which describes the transitive state in the change of potential energy in the system.

Here a new method for expressing the stiffness matrix is introduced. Any twist of the moving platform of a 6-DOF in-parallel mechanism can be decomposed into a linear combination of six pure rotations about the line-vectors representing the six connectors of the mechanism, provided that the mechanism is not in a singular position. The coordinates of the six line-vectors, $\{\hat{s}_1,\hat{s}_2,\hat{s}_3,\hat{s}_4,\hat{s}_5,\hat{s}_6\}$, which are screws $\{\$_1,\$_2,\$_3,\$_4,\$_5,\$_6\}$ of zero pitch, can be determined from the configuration of the 6-DOF in-parallel mechanism. If a stiffness mapping is in its collineation form, the i-th column elements of the stiffness mapping are the components of the force produced by a rotation about the i-th connector chain of the in-parallel mechanism. The dimension is force through the entire matrix. Further, the potential energy in the system can be derived from the collineation form of the stiffness mapping.

This formulation may be used to compare the difference between two mappings. The comparison between two mappings is based on the mappings being represented in the same basis. The basis transition can be made by using the transition matrices between the two bases to transform the second stiffness matrix into the same basis as the first one.

2. Basis Selection

The space of twists can be identified with the Lie algebra, e(3), of the Euclidean group, E(3). e(3) is a six-dimensional vector space and e(3) has, obviously, an infinite number of bases. One basis of e(3), at first very appealing, is the one formed by three rotations and three translations along the coordinate axes, usually perpendicular, of an arbitrarily chosen coordinate system. Here, this basis is called the "initial basis". In dual vector form, the initial basis is given by

$$\{\underline{i},\ \underline{j},\ \underline{k};\ \varepsilon\underline{i},\ \varepsilon\underline{j},\ \varepsilon\underline{k},\} \tag{1}$$

where $\underline{i}$, $\underline{j}$,and $\underline{k}$, are unit vectors along the coordinate axes, and ε represents the dual unit (ε^2=0).

Any twist can be represented by

$$\hat{s} = L\underline{i} + M\underline{j} + N\ \underline{k} + P\varepsilon\underline{i} + Q\varepsilon\underline{j} + R\varepsilon\underline{k}. \tag{2}$$

where (L, M, N; P, Q, R) are the twist coordinates.

It should be recognized, however, that the basis given by (5.1) has several disadvantages. Firstly, it is not orthogonal under the Klein form [49]. Secondly, while the first elements of the basis represent infinitesimal rotations, the last three represent infinitesimal translations. Hence, they have different dimensions.

Fortunately, a 6-DOF in-parallel mechanism provides a "natural" basis for e(3). Indeed, if the mechanism is not in a singular configuration, then the six line vectors along the connectors are linearly independent. If one denotes $\hat{s}_n$ as the normalized line vector along the n-th connector, then

$$\hat{s}_n = \underline{s}_n + \varepsilon\underline{s}_{0n} \ with\ \underline{s}_n \cdot \underline{s}_n = 1 \ and\ \underline{s}_n \cdot \underline{s}_{0n} = 0, \tag{3}$$

and, the "natural" basis will be given by

$$\{\hat{s}_1,\ \hat{s}_2,\ \hat{s}_3,\ \hat{s}_4,\ \hat{s}_5,\ \hat{s}_6\}. \tag{4}$$

Unlike the basis given by equation (1), all twists in the basis given by equation (4) are infinitesimal rotations and, therefore, have the same dimension.

An arbitrary twist $\hat{s} \in e(3)$ can be written as a linear combination of the twists of the natural basis as follows

$$\hat{s} = \lambda_1 \hat{s}_1 + \lambda_2 \hat{s}_2 + \lambda_3 \hat{s}_3 + \lambda_4 \hat{s}_4 + \lambda_5 \hat{s}_5 + \lambda_6 \hat{s}_6, \tag{5}$$

thus

$$\underline{\lambda}^{nat} = [\ \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ ]^T \tag{6}$$

is the coordinate vector of $\hat{s}$ with respect to the "natural" basis given by (4). However, each of the twists in (4) can be written in terms of the initial basis as follows

$$\hat{s}_n = L_n \underline{i} + M_n \underline{j} + N_n \underline{k} + P_n \varepsilon \underline{i} + Q_n \varepsilon \underline{j} + R_n \varepsilon \underline{k}, \quad for\ n=1,2,3,4,5,6. \tag{7}$$

Substituting (7) into (5) and grouping the terms yields

$$\begin{aligned} \underline{\hat{s}} = {} & (\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 + \lambda_4 L_4 + \lambda_5 L_5 + \lambda_6 L_6)\underline{i} + \\ & (\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 + \lambda_4 M_4 + \lambda_5 M_5 + \lambda_6 M_6)\underline{j} + \\ & (\lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 N_3 + \lambda_4 N_4 + \lambda_5 N_5 + \lambda_6 N_6)\underline{k} + \\ & (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4 + \lambda_5 P_5 + \lambda_6 P_6)\varepsilon\underline{i} + \\ & (\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 + \lambda_4 Q_4 + \lambda_5 Q_5 + \lambda_6 Q_6)\varepsilon\underline{j} + \\ & (\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 + \lambda_4 R_4 + \lambda_5 R_5 + \lambda_6 R_6)\varepsilon\underline{k}\ . \end{aligned} \tag{8}$$

The coordinate vector of $\hat{s}$ with respect to the "initial" basis is given by

$$\underline{\lambda}^{ini} = \begin{pmatrix} \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 + \lambda_4 L_4 + \lambda_5 L_5 + \lambda_6 L_6 \\ \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 + \lambda_4 M_4 + \lambda_5 M_5 + \lambda_6 M_6 \\ \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 N_3 + \lambda_4 N_4 + \lambda_5 N_5 + \lambda_6 N_6 \\ \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4 + \lambda_5 P_5 + \lambda_6 P_6 \\ \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 + \lambda_4 Q_4 + \lambda_5 Q_5 + \lambda_6 Q_6 \\ \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 + \lambda_4 R_4 + \lambda_5 R_5 + \lambda_6 R_6 \end{pmatrix} \tag{9}$$

or in matrix form,

$$\underline{\lambda}^{ini} = j\underline{\lambda}^{nat}, \tag{10}$$

where

$$j = \begin{pmatrix} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\ M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\ N_1 & N_2 & N_3 & N_4 & N_5 & N_6 \\ P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\ R_1 & R_2 & R_3 & R_4 & R_5 & R_6 \end{pmatrix}. \tag{11}$$

The matrix j is called the transition matrix from the "natural" basis to the "initial" basis. Given a coordinate vector, $\underline{\lambda}^{nat}$, of ŝ with respect to the "natural" basis, j $\underline{\lambda}^{nat}$ provides the coordinate vector, $\underline{\lambda}^{ini}$, of the same twist ŝ with respect to the "initial" basis. Furthermore, the i-th column of the transition matrix j represents the coordinates of $\hat{s}_i$, the coordinates of the i-th connector, with respect to the "initial" basis.

It is usually shown in standard courses in linear algebra, for example Ref.50, that j is the representative matrix of the identity mapping

$I_{e(3)} : e(3) \rightarrow e(3), \qquad I_{e(3)}(\$) = \$$

on e(3) when two distinct bases are employed to represent the mapping, therefore

a. j is a non-singular matrix, and

b. j^{-1} is the transition matrix from the "initial" basis to the "natural" basis.

3. The Stiffness Mapping in The Natural Basis

A stiffness mapping is represented by a 6x6 matrix [K] and relates an infinitesimal twist, with coordinates $\delta\underline{D}$, to a change in wrench, with coordinates $\delta\underline{w}$. The stiffness mapping can be expressed in the form

$$\delta\underline{w}=[K]\delta\underline{D}\ , \tag{12}$$

where $\delta\underline{w}=[\delta\underline{f};\ \delta\underline{m}_o]$ is in ray coordinates, and $\delta\underline{D}=[\delta\underline{x}_o;\ \delta\underline{\theta}]$ is in axis coordinates. The two 3x1 vectors of the wrench are the small force change $\delta\underline{f}$ and the small moment change $\delta\underline{m}_o$. The two 3x1 vectors of the twist are the small translation $\delta\underline{x}_o$ and the small rotation angle $\delta\underline{\theta}$. The 6x6 matrix [K] (see Chapter 2 for details) is

$$[K]= j[k_i]j^{\,T}+ [\delta j_\theta][k_i(1-\frac{l_{0i}}{l_0})]([\delta j_\theta]^T+[v_\theta]^T)\ +\ [\delta j_\alpha][k_i(1-\frac{l_{0i}}{l_0})]([\delta j_\alpha]^T+[v_\alpha]^T). \tag{13}$$

This matrix represents the "correlation form of stiffness mapping" (see Refs.1 and 13). However in this application, the collineation form of the stiffness is needed for which $\delta\underline{d} = \Delta\delta\underline{D}$ and

$$\delta\underline{w}\ =\ ([K]\Delta)\delta\underline{d}\ =\ [k]\delta\underline{d}\ , \tag{14}$$

where $\delta\underline{d} = [\delta\underline{\theta};\ \delta\underline{x}_o]$ is now in ray coordinates and where

$$\Delta=\begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}. \tag{15}$$

The notation [k]=[K]Δ has been introduced. [k] is the collineation form of stiffness mapping since it maps a twist having ray coordinates $\delta\underline{d}$ into the wrench change having ray coordinates $\delta\underline{w}$.

Consider a general 6-DOF in-parallel mechanism in a position that is *non-singular.* The line-vectors $\{\hat{s}_1,\hat{s}_2,\hat{s}_3, \hat{s}_4,\hat{s}_5,\hat{s}_6\}$ of the six connectors between upper and lower platform are linearly independent (A geometrical singularity of the six connectors occurs when the rank of j < 6. It follows from (13) that this condition will produce a singular value of [K] at the unloaded position $(1-l_{0i}/l_i) = 0$ for which $[K]=j[k_i]j^T$). Thus the set of screws $\{\hat{s}_1,\hat{s}_2,\hat{s}_3,\hat{s}_4, \hat{s}_5,\hat{s}_6\}$, all with zero pitches, is selected as a basis in e(3) for the mechanism in this configuration. In the space of twists, the basis is composed of pure rotations only, and the translation portion of any twist is incorporated into the rotations. In the space of wrenches, the basis is composed of pure forces only, and the couple portion of a wrench is incorporated into the forces. Let the "initial" basis $\{\underline{i}, \underline{j}, \underline{k}, \varepsilon\underline{i}, \varepsilon\underline{j}, \varepsilon\underline{k}\}$ and the "natural" basis $\{\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4, \hat{s}_5, \hat{s}_6\}$ be denoted by "ini" and "nat".

Figures b.1 and b.2 illustrate an instantaneous change in wrench $\delta\underline{w}$ due to an infinitesimal twist $\delta\underline{d}$ that are represented in the bases "ini" and "nat" respectively.

Let $B=\{\$_1, \$_2, \$_3, \$_4, \$_5, \$_6\}$ and $B'=\{\$'_1, \$'_2, \$'_3, \$'_4, \$'_5, \$'_6\}$ be any two bases of e(3). Let $\delta\underline{d}$ be an arbitrary twist, and let $\delta\underline{d}_B$ and $\delta\underline{d}_{B'}$ be the coordinate vectors of the twist $\delta\underline{d}$ with respect to basis B and B' respectively; i.e.

$$\delta\underline{d}_B = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6]^T \tag{16}$$

and

$$\delta\underline{d}_{B'} = [\lambda'_1\ \lambda'_2\ \lambda'_3\ \lambda'_4\ \lambda'_5\ \lambda'_6]^T\ , \tag{17}$$

where

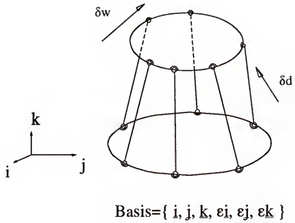

Figure b.1 Screws δw and δd are represented in the dual vector basis A.

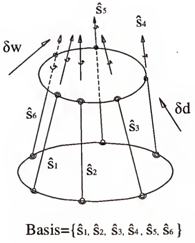

Figure b.2 Screws δw and δd are represented in the selected basis B.

$$\delta \underline{d} = \lambda_1 \$_1 + \lambda_2 \$_2 + \lambda_3 \$_3 + \lambda_4 \$_4 + \lambda_5 \$_5 + \lambda_6 \$_6 \qquad (18)$$

and

$$\delta \underline{d} = \lambda'_1 \$'_1 + \lambda'_2 \$'_2 + \lambda'_3 \$'_3 + \lambda'_4 \$'_4 + \lambda'_5 \$'_5 + \lambda'_6 \$'_6 \qquad (19)$$

Similarly, let $\delta\underline{w}$ be an arbitrary wrench, and let $\delta\underline{w}_{B}$ and $\delta\underline{w}_{B'}$ be the coordinate vectors of the twist $\delta\underline{w}$ with respect to basis B and B' respectively; i.e.

$$\delta \underline{w}_B = [\mu_1\ \mu_2\ \mu_3\ \mu_4\ \mu_5\ \mu_6]^T \qquad (20)$$

and

$$\delta \underline{w}_{B'} = [\mu'_1\ \mu'_2\ \mu'_3\ \mu'_4\ \mu'_5\ \mu'_6]^T \qquad (21)$$

where

$$\delta \underline{w} = \mu_1 \$_1 + \mu_2 \$_2 + \mu_3 \$_3 + \mu_4 \$_4 + \mu_5 \$_5 + \mu_6 \$_6 \qquad (22)$$

and

$$\delta \underline{w} = \mu'_1 \$'_1 + \mu'_2 \$'_2 + \mu'_3 \$'_3 + \mu'_4 \$'_4 + \mu'_5 \$'_5 + \mu'_6 \$'_6 \qquad (23)$$

Furthermore, let $j_{B\text{-}B'}$ be the transition matrix from basis B to basis B'. As mentioned in Section 1, the i-th column of $j_{B\text{-}B'}$ is the coordinate vector of $\$_i$ with respect to the basis B'. Furthermore,

$$(j_{B-B'})^{-1} = j_{B'-B}\ ; \qquad (24)$$

i.e. the inverse of $j_{B\text{-}B'}$ exists and is the transition matrix from basis B' and basis B.

The coordinate vectors of both $\delta\underline{d}$ and $\delta\underline{w}$ are related by

$$\delta\underline{d}_{B'} = j_{B-B'}\,\delta\underline{d}_{B} \tag{25}$$

and

$$\delta\underline{w}_{B'} = j_{B-B'}\,\delta\underline{w}_{B}\ . \tag{26}$$

Finally, assume that $[k_B]$ is the stiffness matrix of a 6-DOF in-parallel platform in collineation form with respect to basis B; i.e.

$$\delta\underline{w}_{B} = [k_B]\ \delta\underline{d}_{B}\ . \tag{27}$$

Then, the stiffness matrix of the same 6-DOF in-parallel platform (in the same relative position of the upper platform with respect to the lower platform) in collineation form with respect to basis B' is obtained by substituting $\delta\underline{d}_B$ and $\delta\underline{w}_B$, in equation (27), in terms of $\delta\underline{d}_{B'}$ and $\delta\underline{w}_{B'}$, as follows

$$(j_{B-B'})^{-1}\delta\underline{w}_{B'} = [k_B](j_{B-B'})^{-1}\delta\underline{d}_{B'}\ . \tag{28}$$

The pre-multiplication of both sides of equation (28) by $j_{B-B'}$ yields

$$\delta\underline{w}_{B'} = j_{B-B'}\ [k_B](j_{B-B'})^{-1}\delta\underline{d}_{B'}\ . \tag{29}$$

Thus, the sought stiffness matrix is

$$[K_{B'}]\Delta = j_{B-B'}\ [k_B](j_{B-B'})^{-1}\ . \tag{30}$$

This result is for any two arbitrary bases of e(3), in particular, it is valid for the initial and natural bases discussed in Section 2. Further, the application of the result to the initial and natural bases, together with a dimensional analysis of the resulting formula, proves the homogeneity of the stiffness matrix with respect to the natural basis.

If the transition matrices, and the stiffness matrix in collineation form with respect to the initial basis are written using 3x3 submatrices, each one of these submatrices is dimensionally homogenous, and from equation (11)

$$j = \begin{pmatrix} 1 & 1 \\ L & L \end{pmatrix} , \quad j^{-1} = \begin{pmatrix} 1 & L^{-1} \\ 1 & L^{-1} \end{pmatrix}, \tag{31}$$

where 1 stands for an adimensional 3x3 submatrix. From equation (5.14)

$$[k_{nat}] = j^{-1}[k_{ini}]j = \begin{pmatrix} 1 & L^{-1} \\ 1 & L^{-1} \end{pmatrix} \begin{pmatrix} F & FL^{-1} \\ FL & F \end{pmatrix} \begin{pmatrix} 1 & 1 \\ L & L \end{pmatrix} = \begin{pmatrix} F & F \\ F & F \end{pmatrix} . \tag{32}$$

Hence, it has been shown that the stiffness mapping is represented in collineation form, with respect to the natural basis, by a dimensionally homogenous 6x6 matrix.

A similar analysis shows that the stiffness mapping is represented in collineation form by a dimensionally homogenous 6x6 matrix, provided that all of the elements of the basis satisfy

$$Ki(\$_i,\$_i) \neq 0 \quad (for\ i=1,2,3,4,5,6), \tag{33}$$

where Ki stands for the Killing form of the Lie algebra of e(3); i.e. provided that the screws are not at the plane at infinity.

Consider now a small rotation $\delta\varphi_n$ of the platform about a connector line $\hat{s}_n$. This produces a change in wrench

$$\delta \underline{w}_n = [k_{nat}]\ \delta\varphi_n\ \hat{s}_n\ , \tag{34}$$

where $[k_{nat}]=[K_{nat}]\Delta$, the stiffness matrix in collineation form ,with respect to the natural basis, has been introduced. Here the rotation is expressed in the form, $0\hat{s}_1 + 0\hat{s}_2 +....+ 0\hat{s}_{n-1} + \delta\varphi_n\hat{s}_n + 0\hat{s}_{n+1} +....+ 0\hat{s}_6$. This rotation produces displacements $\sigma_{1n}, \sigma_{2n}, \sigma_{3n}, \sigma_{4n}, \sigma_{5n}, \sigma_{6n}$ in each of the six connectors. Therefore, the wrench produced by the stiffness mapping due to the small rotation about the connector line $\hat{s}_n$ is

$$\delta\underline{w}_n = k_1\sigma_{1n}\hat{s}_1 + k_2\sigma_{2n}\hat{s}_2 + k_3\sigma_{3n}\hat{s}_3 + k_4\sigma_{4n}\hat{s}_4 + k_5\sigma_{5n}\hat{s}_5 + k_6\sigma_{6n}\hat{s}_6 \,, \tag{35}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are the spring stiffnesses of the connectors.

Consider now a twist given by

$$\delta\underline{d} = \delta\varphi_1\hat{s}_1 + \delta\varphi_2\hat{s}_2 + \delta\varphi_3\hat{s}_3 + \delta\varphi_4\hat{s}_4 + \delta\varphi_5\hat{s}_5 + \delta\varphi_{6},\hat{s}_6 \tag{36}$$

then, since the stiffness mapping is a linear one,

$$\begin{aligned}\delta\underline{w} = \sum_{n=1}^{6}\delta w_n = {} & k_1\sigma_{11}\hat{s}_1 + k_2\sigma_{21}\hat{s}_2 + k_3\sigma_{31}\hat{s}_3 + k_4\sigma_{41}\hat{s}_4 + k_5\sigma_{51}\hat{s}_5 + k_6\sigma_{61}\hat{s}_6 \\ & + k_1\sigma_{12}\hat{s}_1 + k_2\sigma_{22}\hat{s}_2 + k_3\sigma_{32}\hat{s}_3 + k_4\sigma_{42}\hat{s}_4 + k_5\sigma_{52}\hat{s}_5 + k_6\sigma_{62}\hat{s}_6 \\ & + k_1\sigma_{13}\hat{s}_1 + k_2\sigma_{23}\hat{s}_2 + k_3\sigma_{33}\hat{s}_3 + k_4\sigma_{43}\hat{s}_4 + k_5\sigma_{53}\hat{s}_5 + k_6\sigma_{63}\hat{s}_6 \\ & + k_1\sigma_{14}\hat{s}_1 + k_2\sigma_{24}\hat{s}_2 + k_3\sigma_{34}\hat{s}_3 + k_4\sigma_{44}\hat{s}_4 + k_5\sigma_{54}\hat{s}_5 + k_6\sigma_{64}\hat{s}_6 \\ & + k_1\sigma_{15}\hat{s}_1 + k_2\sigma_{25}\hat{s}_2 + k_3\sigma_{35}\hat{s}_3 + k_4\sigma_{45}\hat{s}_4 + k_5\sigma_{55}\hat{s}_5 + k_6\sigma_{65}\hat{s}_6 \\ & + k_1\sigma_{16}\hat{s}_1 + k_2\sigma_{26}\hat{s}_2 + k_3\sigma_{36}\hat{s}_3 + k_4\sigma_{46}\hat{s}_4 + k_5\sigma_{56}\hat{s}_5 + k_6\sigma_{66}\hat{s}_6 .\end{aligned} \tag{37}$$

or

$$\begin{aligned}\delta\underline{w} = {} & k_1(\sigma_{11} + \sigma_{12} + \sigma_{13} + \sigma_{14} + \sigma_{15} + \sigma_{16})\ \hat{s}_1 + \\ & + k_2(\sigma_{21} + \sigma_{22} + \sigma_{23} + \sigma_{24} + \sigma_{25} + \sigma_{26})\ \hat{s}_2 \\ & + k_3(\sigma_{31} + \sigma_{32} + \sigma_{33} + \sigma_{34} + \sigma_{35} + \sigma_{36})\ \hat{s}_3 \\ & + k_4(\sigma_{41} + \sigma_{42} + \sigma_{43} + \sigma_{44} + \sigma_{45} + \sigma_{46})\ \hat{s}_4 \\ & + k_5(\sigma_{51} + \sigma_{52} + \sigma_{53} + \sigma_{54} + \sigma_{55} + \sigma_{56})\ \hat{s}_5 \\ & + k_6(\sigma_{61} + \sigma_{62} + \sigma_{63} + \sigma_{64} + \sigma_{65} + \sigma_{66})\ \hat{s}_6 .\end{aligned} \tag{38}$$

Therefore the overall induced wrench by the six rotations is

$$\begin{aligned}\delta \underline{w} = \sum_{n=1}^{6} \delta \underline{w}_n = {} & k_1\sigma_{11}\hat{s}_1 + k_2\sigma_{21}\hat{s}_2 + k_3\sigma_{31}\hat{s}_3 + k_4\sigma_{41}\hat{s}_4 + k_5\sigma_{51}\hat{s}_5 + k_6\sigma_{61}\hat{s}_6 \\ & + k_1\sigma_{12}\hat{s}_1 + k_2\sigma_{22}\hat{s}_2 + k_3\sigma_{32}\hat{s}_3 + k_4\sigma_{42}\hat{s}_4 + k_5\sigma_{52}\hat{s}_5 + k_6\sigma_{62}\hat{s}_6 \\ & + k_1\sigma_{13}\hat{s}_1 + k_2\sigma_{23}\hat{s}_2 + k_3\sigma_{33}\hat{s}_3 + k_4\sigma_{43}\hat{s}_4 + k_5\sigma_{53}\hat{s}_5 + k_6\sigma_{63}\hat{s}_6 \\ & + k_1\sigma_{14}\hat{s}_1 + k_2\sigma_{24}\hat{s}_2 + k_3\sigma_{34}\hat{s}_3 + k_4\sigma_{44}\hat{s}_4 + k_5\sigma_{54}\hat{s}_5 + k_6\sigma_{64}\hat{s}_6 \\ & + k_1\sigma_{15}\hat{s}_1 + k_2\sigma_{25}\hat{s}_2 + k_3\sigma_{35}\hat{s}_3 + k_4\sigma_{45}\hat{s}_4 + k_5\sigma_{55}\hat{s}_5 + k_6\sigma_{65}\hat{s}_6 \\ & + k_1\sigma_{16}\hat{s}_1 + k_2\sigma_{26}\hat{s}_2 + k_3\sigma_{36}\hat{s}_3 + k_4\sigma_{46}\hat{s}_4 + k_5\sigma_{56}\hat{s}_5 + k_6\sigma_{66}\hat{s}_6\end{aligned}$$

or

$$\begin{aligned}\delta \underline{w} = {} & k_1(\sigma_{11} + \sigma_{12} + \sigma_{13} + \sigma_{14} + \sigma_{15} + \sigma_{16})\ \hat{s}_1 + \\ & + k_2(\sigma_{21} + \sigma_{22} + \sigma_{23} + \sigma_{24} + \sigma_{25} + \sigma_{26})\ \hat{s}_2 \\ & + k_3(\sigma_{31} + \sigma_{32} + \sigma_{33} + \sigma_{34} + \sigma_{35} + \sigma_{36})\ \hat{s}_3 \\ & + k_4(\sigma_{41} + \sigma_{42} + \sigma_{43} + \sigma_{44} + \sigma_{45} + \sigma_{46})\ \hat{s}_4 \\ & + k_5(\sigma_{51} + \sigma_{52} + \sigma_{53} + \sigma_{54} + \sigma_{55} + \sigma_{56})\ \hat{s}_5 \\ & + k_6(\sigma_{61} + \sigma_{62} + \sigma_{63} + \sigma_{64} + \sigma_{65} + \sigma_{66})\ \hat{s}_6.\end{aligned} \tag{39}$$

Thus, it is shown that a twist given by equation (36) produces the wrench given above, on the other hand

$$\delta \underline{w}_{nat} = [k_{nat}]\ \delta \underline{d}_{nat}\ . \tag{40}$$

Thus,

$$\begin{pmatrix} k_1(\sigma_{11}+\sigma_{12}+\ldots\ldots+\sigma_{16}) \\ k_2(\sigma_{21}+\sigma_{22}+\ldots\ldots+\sigma_{26}) \\ k_3(\sigma_{31}+\sigma_{32}+\ldots\ldots+\sigma_{36}) \\ k_4(\sigma_{41}+\sigma_{42}+\ldots\ldots+\sigma_{46}) \\ k_5(\sigma_{51}+\sigma_{52}+\ldots\ldots+\sigma_{56}) \\ k_6(\sigma_{61}+\sigma_{62}+\ldots\ldots+\sigma_{66}) \end{pmatrix} = [k_{nat}\ ]\delta \underline{d}_{nat}. \tag{41}$$

Assume that $[k_{nat}]$ is the stiffness matrix, in collineation form, with respect to the natural basis and that $\delta\varphi_1 = \delta\varphi_2 = \delta\varphi_3 = \delta\varphi_4 = \delta\varphi_5 = \delta\varphi_6 = 1$.

Consider that $\delta\underline{d} = 1\hat{s}_1 + 0\hat{s}_2 + 0\hat{s}_3 + 0\hat{s}_4 + 0\hat{s}_5 + 0\hat{s}_6$, then the coordinate system of the resulting wrench, with respect to the natural basis is given by:

1. Using the stiffness matrix

$$[k_{nat}]\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}k_{nat}[1,1]\\k_{nat}[2,1]\\k_{nat}[3,1]\\k_{nat}[4,1]\\k_{nat}[5,1]\\k_{nat}[6,1]\end{bmatrix}.$$

2. Using equation (36) yields

$$\delta\underline{w}_n\,|_{\delta\underline{d}=1\hat{s}_1} = k_1\sigma_{11}\hat{s}_1 + k_2\sigma_{21}\hat{s}_2 + k_3\sigma_{31}\hat{s}_3 + k_4\sigma_{41}\hat{s}_4 + k_5\sigma_{51}\hat{s}_5 + k_6\sigma_{61}\hat{s}_6\ , \tag{42}$$

therefore, the coordinate vector is $[k_1\sigma_{11}\ k_2\sigma_{21}\ k_3\sigma_{31}\ k_4\sigma_{41}\ k_5\sigma_{51}\ \ k_6\sigma_{61}\ ]^T$. The stiffness matrix in the natural basis is given in the form,

$$[k_{nat}] = \begin{pmatrix} k_1\sigma_{11} & k_1\sigma_{12} & k_1\sigma_{13} & k_1\sigma_{14} & k_1\sigma_{15} & k_1\sigma_{16} \\ k_2\sigma_{21} & k_2\sigma_{22} & k_2\sigma_{23} & k_2\sigma_{24} & k_2\sigma_{25} & k_2\sigma_{26} \\ k_3\sigma_{31} & k_3\sigma_{32} & k_3\sigma_{33} & k_3\sigma_{34} & k_3\sigma_{35} & k_3\sigma_{36} \\ k_4\sigma_{41} & k_4\sigma_{42} & k_4\sigma_{43} & k_4\sigma_{44} & k_4\sigma_{45} & k_4\sigma_{46} \\ k_5\sigma_{51} & k_5\sigma_{52} & k_5\sigma_{53} & k_5\sigma_{54} & k_5\sigma_{55} & k_5\sigma_{56} \\ k_6\sigma_{61} & k_6\sigma_{62} & k_6\sigma_{63} & k_6\sigma_{64} & k_6\sigma_{65} & k_6\sigma_{66} \end{pmatrix}. \tag{43}$$

It should be noted that the space of wrenches can be transformed from the natural basis to the initial basis.

The potential energy of the system can be given by,

$$P.E. = \sum_{m=1}^{6} k_m \left(\sum_{n=1}^{6} \sigma_{mn}\right)^2 \tag{44}$$

4. Symmetric Stiffness Mapping

From (13), it is interesting to note that if the platform is in the unloaded configuration, then

$$[K] = [K_{ini}]_o = j\,[k_i]\,j^{T}. \tag{45}$$

Thus, the collineation form of $[K_{ini}]_o$ after a change of basis is in the form,

$$[k_{nat}]_o = j^{-1}\,[K_{nat}]_o \Delta\, j = j^{-1} j\,[k_i]\,j^{T} \Delta j = [k_i]\,j^{T}\,\Delta j. \tag{46}$$

Now

$$j^{T} = \begin{pmatrix} \underline{s}_1^{T}, \underline{s}_{01}^{\;T} \\ \underline{s}_2^{T}, \underline{s}_{02}^{\;T} \\ \underline{s}_3^{T}, \underline{s}_{03}^{\;T} \\ \underline{s}_4, \underline{s}_{04}^{\;T} \\ \underline{s}_5^{T}, \underline{s}_{05}^{\;T} \\ \underline{s}_6^{T}, \underline{s}_{06}^{\;T} \end{pmatrix} \quad and \quad \Delta j = \begin{pmatrix} \underline{s}_{01} & \underline{s}_{02} & \underline{s}_{03} & \underline{s}_{04} & \underline{s}_{05} & \underline{s}_{06} \\ \underline{s}_1 & \underline{s}_2 & \underline{s}_3 & \underline{s}_4 & \underline{s}_5 & \underline{s}_6 \end{pmatrix}$$

thus

$$j^{T}\Delta j = \begin{pmatrix} 0 & \underline{s}_1^{T}\underline{s}_{02} + \underline{s}_2^{T}\underline{s}_{01} & \cdots\cdots & \underline{s}_1^{T}\underline{s}_{06} + \underline{s}_6^{T}\underline{s}_{01} \\ \underline{s}_2^{T}\underline{s}_{01} + \underline{s}_1^{T}\underline{s}_{02} & 0 & \cdots\cdots & \underline{s}_2^{T}\underline{s}_{06} + \underline{s}_6^{T}\underline{s}_{02} \\ \vdots & \vdots & \vdots & \\ \underline{s}_6^{T}\underline{s}_{01} + \underline{s}_1^{T}\underline{s}_{06} & \underline{s}_6^{T}\underline{s}_{02} + \underline{s}_2^{T}\underline{s}_{06} & \cdots & 0 \end{pmatrix}$$

or

$$j^T \Delta j = \begin{pmatrix} 0 & m_{12} & m_{13} & \dots & m_{16} \\ m_{12} & 0 & m_{23} & \dots & m_{26} \\ m_{13} & m_{23} & 0 & \dots & m_{36} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{16} & m_{26} & m_{36} & \dots & 0 \end{pmatrix},$$

where the elements m_{ij} are the mutual moments of the connector lines (symmetric with diagonal elements zero).

5. Comparison of Stiffness Mappings

Since the dimension is the same in every element of the stiffness matrix, a meaningful comparison for a set of stiffness mappings may be established. The computation of the potential energy in the elastic system is as described in a prior section. The implementation yields a unique parameter to compare the given mappings. It always yields an answer for comparison. An example is presented in the next section to illustrate this behavior.

6. Numerical Example

The example is consider a 6-DOF in-parallel platform is shown in Figure 2.5 (see Sec.2.4). The movable platform is connected to the ground via six translational springs (considered as prismatic joints) acting in-parallel. The six springs define a spatial six-dimensional elastic system. Each connector is considered as a sequence of spherical-prismatic-spherical joints.

The movable platform is defined by its spherical pivot points o, p, q, r, s, and t which are sub-labeled with 0 and 1 to distinguish bottom and top, respectively. A coordinate system is fixed to the ground, with its origin at o_0. Point p_0, s_0 is on the *x*-axis, while points q_0, r_0, t_0 lie in the *xy* plane. All points are located relative to o_0.

$\|o_0p_0\|=\|p_0q_0\|=\|q_0o_0\|=$ 4.500", and $\|o_0s_0\|=\|p_0t_0\|=\|q_0r_0\|=$ 5.400" are lengths between pivot points on the bottom plate. $\|r_1s_1\|=\|s_1t_1\|=\|t_1r_1\|=$ 4.500", and $\|r_1o_1\|=\|s_1p_1\|=\|t_1q_1\|=$ 3.600", are lengths between pivot points on the top plate. At its nominal position, the connector lengths for connectors $\|o_0o_1\|$, $\|p_0p_1\|$, $\|q_0q_1\|$ are 3.6990", and for connectors $\|s_0s_1\|$, $\|t_0t_1\|$, $\|r_0r_1\|$ are 3.2323", and the coordinates of the geometric center point c of the upper triangle is (2.2500, 1.2990, 2.4345).

The stiffness matrix [K] can be derived by using line geometry. Suppose that the all of the spring stiffnesses of connectors of the in-parallel mechanism are 40.0lb/in. The platform is rotated 0.5 and -2.0 degrees about z-axis with respect to point c from its nominal position separately, i.e. first it is moved 0.5 degree, then brought back to the nominal position, and then moved -2.0 degrees. Let the initial basis be denoted as A; the other bases are such that the selected basis for the platform at the specified position: the natural bases for the movable platform at the nominal position, the selected basis for the position of the platform after 0.5 degree rotation about z-axis, and the selected basis for the position of the platform after -2.0 degree rotation about z-axis, are denoted as B_1, B_2, and B_3.

These [K]s are ready to be calculated by using equation (13). The [K]s and js matrices are simulated by using C language on a Personal Computer, and analyzed in Mathcad. $[K_{nom}]$ and j_{nom} represented in the initial basis for the movable platform at its nominal position are,

$$[K_{nom}]=\begin{bmatrix} 59.9762 & 0 & 0 & 26.9854 & 124.2725 & -77.9114 \\ 0 & 59.9762 & 0 & -124.2725 & 26.9854 & 134.9465 \\ 0 & 0 & 120.0475 & 155.9463 & -270.1069 & -53.9708 \\ 26.9854 & -124.2725 & 155.9463 & 773.1584 & -350.8792 & -384.7783 \\ 124.2725 & 26.9854 & -270.1069 & -350.8792 & 1.1783\cdot 10^{3} & 20.7168 \\ -77.9114 & 134.9465 & -53.9708 & -384.7783 & 20.7168 & 949.0514 \end{bmatrix} \quad j_{nom}=\begin{bmatrix} 0.7402 & -0.5849 & -0.2508 & 0.5532 & -0.4894 & 0.0317 \\ -0.1378 & -0.3011 & 0.7099 & -0.3559 & -0.5721 & 0.6571 \\ 0.6581 & 0.7532 & 0.6581 & 0.7532 & 0.6581 & 0.7532 \\ 0 & 0 & 0 & 3.5223 & 2.5648 & -0.587 \\ 0 & -4.0672 & -2.9615 & -1.3557 & -1.4808 & 0.3389 \\ 0 & -1.626 & 3.1947 & -3.2278 & 0.6201 & -0.271 \end{bmatrix}$$

$[K_{0.5}]$ and $j_{0.5}$ are

$$[K_{0.5}]=\begin{bmatrix} 59.6371 & 0 & 0 & 27.5364 & 123.9401 & -77.4709 \\ 0 & 59.6371 & 0 & -123.9401 & 27.5364 & 134.1835 \\ 0 & 0 & 120.3348 & 156.3196 & -270.7534 & -55.0728 \\ 27.5364 & -124.4159 & 156.3196 & 775.8615 & -349.36 & -387.2482 \\ 124.4159 & 27.5364 & -270.7534 & -354.078 & 1.182\cdot 10^{3} & 24.2497 \\ -77.4709 & 134.1835 & -55.0728 & -386.1777 & 24.8678 & 939.6038 \end{bmatrix} \quad j_{0.5}=\begin{bmatrix} 0.7422 & -0.5814 & -0.2531 & 0.5497 & -0.4891 & 0.0317 \\ -0.1362 & -0.2991 & 0.7109 & -0.354 & -0.5747 & 0.653 \\ 0.6562 & 0.7567 & 0.6562 & 0.7567 & 0.6562 & 0.7567 \\ 0 & 0 & 0 & 3.5385 & 2.5571 & -0.5898 \\ 0 & -4.0859 & -2.9527 & -1.362 & -1.4763 & 0.3405 \\ 0 & -1.6149 & 3.1991 & -3.2079 & 0.613 & -0.2692 \end{bmatrix}$$

$[K_{-2}]$ and j_{-2} are

$$[K_{-2}]=\begin{bmatrix} 61.2247 & 0 & 0 & 24.628 & 125.4979 & -79.5332 \\ 0 & 61.2247 & 0 & -125.4979 & 24.628 & 137.7555 \\ 0 & 0 & 119.0319 & 154.6269 & -267.8217 & -49.2561 \\ 24.628 & -123.6949 & 154.6269 & 762.9476 & -357.6432 & -374.2918 \\ 123.6949 & 24.628 & -267.8217 & -338.1779 & 1.1647\cdot 10^{3} & 5.5549 \\ -79.5332 & 137.7555 & -49.2561 & -378.3486 & 3.2127 & 983.6869 \end{bmatrix} \quad j_{-2}=\begin{bmatrix} 0.7318 & -0.5985 & -0.2414 & 0.5664 & -0.4905 & 0.032 \\ -0.1438 & -0.3085 & 0.7057 & -0.364 & -0.5619 & 0.6725 \\ 0.6661 & 0.7394 & 0.6661 & 0.7394 & 0.6661 & 0.7394 \\ 0 & 0 & 0 & 3.4577 & 2.5959 & -0.5763 \\ 0 & -3.9926 & -2.9975 & -1.3309 & -1.4988 & 0.3327 \\ 0 & -1.666 & 3.1757 & -3.3041 & 0.6473 & -0.2777 \end{bmatrix}$$

A change of basis of screws from basis A to basis B_1 for $[K_{nom}]$ results into $[k^*_{nom}]= j_{nom}^{-1}[K_{nom}]\Delta j_{nom}$, where

$$[k^*_{nom}]=\begin{bmatrix} 3.3029\cdot 10^{-5} & -20.3858 & 100.4236 & 26.7897 & 100.4236 & -26.3835 \\ -20.3858 & -1.7198\cdot 10^{-5} & -26.3834 & -154.3943 & 26.7899 & -154.3943 \\ 100.4234 & -26.3835 & -3.5469\cdot 10^{-5} & -20.3858 & 100.4235 & 26.7898 \\ 26.7898 & -154.3942 & -20.3857 & 6.4569\cdot 10^{-5} & -26.3834 & -154.3942 \\ 100.4234 & 26.7898 & 100.4234 & -26.3834 & 1.315\cdot 10^{-5} & -20.3859 \\ -26.3833 & -154.3944 & 26.7898 & -154.3944 & -20.3858 & -5.8081\cdot 10^{-5} \end{bmatrix}$$

The diagonal elements of the $[k^*_{nom}]$ are all small numbers, but not zero, due to the truncated errors in computation. It is no problem to be considered as zero since they are small enough relative to the others in the matrix. The result is expected as in the derivation.

Similarly, a change of basis of screws from basis A to basis B_2 for $[K_{0.5}]$ yields $[k^*_{0.5}]=j_{0.5}^{-1}[K_{0.5}]\Delta j_{0.5}$ and a change of basis of screws from basis A to basis B_3 for $[K_{-2}]$ has $[k^*_{-2}]= j_{-2}^{-1}[K_{-2}]\Delta j_{-2}$, where

$$[k^*_{0.5}]=\begin{bmatrix} 0.073 & -20.2414 & 98.8182 & 28.0306 & 99.5911 & -26.6748 \\ -19.575 & -0.0729 & -26.7347 & -154.1656 & 28.5183 & -154.3893 \\ 99.5913 & -26.6748 & 0.0731 & -20.2414 & 98.8182 & 28.0307 \\ 28.5185 & -154.3892 & -19.575 & -0.0729 & -26.7348 & -154.1657 \\ 98.8182 & 28.0307 & 99.5912 & -26.6748 & 0.0729 & -20.2415 \\ -26.7348 & -154.1657 & 28.5184 & -154.3893 & -19.575 & -0.073 \end{bmatrix}$$

$$[k^*_{-2}]=\begin{bmatrix} -0.3086 & -21.2141 & 106.7613 & 21.4801 & 103.5062 & -25.1285 \\ -23.4903 & 0.3085 & -24.848 & -154.9765 & 19.9585 & -153.8035 \\ 103.5066 & -25.1284 & -0.3083 & -21.2141 & 106.7615 & 21.4801 \\ 19.9585 & -153.8034 & -23.4902 & 0.3085 & -24.8481 & -154.9765 \\ 106.7613 & 21.4801 & 103.5064 & -25.1285 & -0.3086 & -21.2141 \\ -24.8481 & -154.9765 & 19.9584 & -153.8034 & -23.4903 & 0.3085 \end{bmatrix}$$

Further, a change of basis for $[k^*_{0.5}]$ and $[k^*_{-2}]$ to be represented in the basis B_1 yield

$$[k^*_{0.5}]_{B_1}=\begin{bmatrix} -0.365 & -20.8968 & 98.9688 & 25.4713 & 99.7369 & -26.9074 \\ -19.5945 & 0.365 & -26.5423 & -154.3162 & 27.8699 & -154.5349 \\ 99.7367 & -26.9074 & -0.3651 & -20.8967 & 98.9687 & 25.4714 \\ 27.8698 & -154.5348 & -19.5943 & 0.3651 & -26.5424 & -154.3161 \\ 98.9686 & 25.4714 & 99.7368 & -26.9073 & -0.3651 & -20.8968 \\ -26.5423 & -154.3163 & 27.8697 & -154.535 & -19.5944 & 0.365 \end{bmatrix}$$

$$[k^*_{-2}]_{B_1}=\begin{bmatrix} 1.3537 & -18.6155 & 106.2551 & 31.5707 & 103.1331 & -24.3702 \\ -23.3907 & -1.3537 & -25.5846 & -154.4702 & 22.7549 & -153.43 \\ 103.133 & -24.3702 & 1.3536 & -18.6155 & 106.255 & 31.5708 \\ 22.7548 & -153.43 & -23.3906 & -1.3536 & -25.5847 & -154.4701 \\ 106.255 & 31.5707 & 103.133 & -24.3701 & 1.3537 & -18.6156 \\ -25.5846 & -154.4703 & 22.7548 & -153.4302 & -23.3906 & -1.3537 \end{bmatrix}$$

The potential energy represented in the basis B_1 for $[k^*_{nom}]_{B1}$, $[k^*_{0.5}]_{B1}$, and $[k^*_{-2}]_{B1}$ are 1.056×10^4lb·in, 1.064×10^4 lb·in, 1.0266×10^4lb·in. The potential energy difference for

$[k^*_{nom}]_{B1}$, $[k^*_{0.5}]_{B1}$ is $\delta P.E_{nom,\,0.5} = |1.056x10^4 - 1.064x10^4| = 0.008x10^4$ lb·in, and for $[k^*_{nom}]_{B1}$, $[k^*_{-2}]_{B1}$ is $\delta P.E_{nom,-2} = |1.056x10^4 - 1.0266x10^4| = 0.0294x10^4$lb·in. Therefore it can be concluded that the stiffness mapping $[K_{0.5}]$ is closer to $[K_{nom}]$ than $[K_{-2}]$.

7. Conclusions

A change of basis for screws which defines a new interpretation for a stiffness mapping is introduced. The interpretation reveals that every element of a given stiffness mapping has the same dimension F (force). Hence, the potential energy for the elastic system could be derived using the elements of the stiffness matrix. This representation provides a viable means to compare the differences between the given stiffness mappings, and it provides physical insight into both the structure of geometric mappings and the concept of comparison.

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